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A Diagram Technique for the Antiferromagnetic Ground-State.

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Summary. — Perturbation series are obtained for the ground-state and energy of two- and three-dimensional lattices with an isotropic antiferromagnetic spin Hamiltonian for general spin S, and with cyclic boundary conditions. Only lattices divisible into two interpenetrating sublattices are considered. The off-diagonal part is treated as the perturbation, and, by introducing spin creation and annihilation operators, the series is evaluated using a diagram technique. No approximations are made. The method can easily be adapted to the anisotropic case. The two degenerate unperturbed ground-states are the symmetric and antisymmetric combinations of the two oppositely oriented completely ordered states. Reasonably good convergence, up to fourth order at least, is obtained for each lattice considered for all S, and numeral calculations for the energy are made for $S=\frac{1}{2}$ and 1. The values are lower than those of previous treatments for all cases except the b.c.c. with $S=\frac{1}{2}$. The general convergence of the series has, however, been assumed, and the results, together with their implication of order in the ground-state, rest therefore upon this supposition. Special methods are developed in the diagram summation for treating (1) the double degeneracy of the unperturbed ground-state, and (2) the apparent N-body divergences in this system where the basic states cannot be represented as collections of « free particles », as in the boson or fermion case, and where the commutator of the annihilation and creation operators is not a c-number. These techniques may be useful for similar problems in other fields.

1. – Introduction.

The two main approaches to the problem of determining the ground-state of the two- and three-dimensional antiferromagnetic lattice are the variational calculations developed particularly by MARSHALL [1], and the spin-wave theory of ANDERSON, KUBO and others, which is summarized in an article by VAN KRANENDONK and VAN VLECK [2].

The system considered is a regular crystal lattice of N atoms with a vector spin $\boldsymbol{S}_i(S_i^x, S_i^y, S_i^z)$ of magnitude $S^{\frac{1}{2}}(S+1)^{\frac{1}{2}}$ associated [3] with each lattice site i, where the axes are oriented in the same way at each i. S_i^z has the $(2S+1)$ eigenvalues $-S \ldots S$, and a complete set of $(2S+1)^N$ orthonormal states define the representation in spin-space, each specified by a set of eigenvalues of the S_i^z, $i=1, 2 \ldots N$. The isotropic exchange Hamiltonian

$$H = J \sum_{\langle ij \rangle} \boldsymbol{S}_i \cdot \boldsymbol{S}_j \tag{1}$$

is assumed, where the sum goes over all nearest-neighbour (n.n.) pairs, and where the exchange integral J is positive and has the same value for each pair. In addition, the treatments of the above authors are confined to those lattices that can be divided into two sublattices A and B, such that all the n.n. of A lie in B and vice versa, though this restriction is not essential for the spin-wave theory.

Since, for a chosen z-axis, the total z-component of spin $S^z = \sum_i S_i^z$ commutes with H, the ground-state of the system can be assumed to have $S^z = 0$. Thus the separate z-components of spin of the two sublattices must be locked equal and opposite, and one important question, *as yet solved only in the one-dimensional case*, is whether the sublattice spin is greater than zero or not, *i.e.*, whether the ground-state has some order or not. MARSHALL considers the case $S = \frac{1}{2}$, and, while not settling the question of order, comes to the conclusion that the ground-state energy lies very close to that of disordered states. NAGAI [4], however, notes that with the ground-state obtained by this treatment, the proportion of anti-parallel pairs decreases with increasing co-ordination number Z. This result, contrary to expectation, leads him to conclude that the disordering effect of the off-diagonal part of (1) is overestimated in Marshall's method.

In the spin-wave theory, which deals with general S, a transformation is made so that H is re-expressed in terms of normal co-ordinates which are linear combinations of the $\boldsymbol{S}_i$, representing stationary « waves of spin » extended

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(1) W. MARSHALL: *Proc. Roy. Soc.* (*London*), A **232**, 48 (1955).
(2) J. VAN KRANENDONK and J. H. VAN VLECK: *Rev. Mod. Phys.*, **30**, 1 (1958).
(3) Here and henceforth we put $\hbar = 1$.
(4) I. NAGAI: *Mem. Fac. Sci. Kyusu Univ.*, B **2**, no. 4, 129 (1958).

through the whole lattice. Certain terms which represent *a*) the dynamical interaction between the spin-waves, and *b*) the fact that the eigenvalues of the S_i^z cannot be greater than S, are neglected, and the Hamiltonian then appears as a sum of independent harmonic oscillators. The zero-point energy of these oscillators is taken as a good approximation to the ground-state energy. This treatment, therefore, in neglecting these terms, implicitly makes the assumption that the ground-state is ordered, and in fact it can be demonstrated ([2]) that the spin-wave ground-state is about 93 % completely ordered, *i.e.* the sublattice z-component of spin is 93 % of its maximum value. MARSHALL has criticised the spin-wave procedure, and points out that for small S the disordering effect of the neglected terms may well be large enough to invalidate the assumptions of order.

In this paper the ground-state and its energy, for the case of general S, will be calculated by perturbation theory in the spin-representation in which (1) is defined, by regarding the off-diagonal part of H as a perturbation and considering one of the two ground-states of the diagonal part, *i.e.* a completely ordered state, as a first approximation. This approach, therefore, makes the assumption that the ground-state is ordered, as does the spin-wave theory, but it yields a perturbation series in which the effects *a*) and *b*) are included. The method is developed in Section 2-Section 5, and explicit calculations are made to fourth order in the energy series in Section 6. It is found that the convergence is reasonably good for the simple cubic (s.c.) and body-centred cubic (b.c.c.) lattices for all S. A discussion of all the results is given in Section 7.

The use of perturbation theory for the antiferromagnetic system gives rise to an N-body problem precisely as in the case of a fermion system described in momentum space, as discussed by GOLDSTONE ([5]), where N here represents the number of spin-sites instead of the number of energy states within the Fermi shell. Creation and annihilation operators for the z-component of spin are constructed for each site i, corresponding to the creation and annihilation operators of momentum, and with this formalism the perturbation series is summed using a diagram technique. There are, however, two important differences from the fermion case that are connected with each other but have to be taken into account separately:

i) The ground-state of the diagonal part of (1) is doubly degenerate, *i.e.* the completely ordered state is degenerate with the state obtained from it by reversing completely the z-component of spin at each site. The « vacuum » state must therefore be defined as the symmetric or antisymmetric combination of these two states. There is no parallel to this in the fermion Hamiltonian discussed by GOLDSTONE.

([5]) J. GOLDSTONE: *Proc. Roy. Soc.* (*London*), A **239**, 267 (1957).

ii) The value of the z-component of spin at a given site, which can take $(2S+1)$ possible values, is the parallel of the occupation number of an energy state in the fermion system which is allowed only 2 values. This difference is reflected by a difference between the two systems in the form of the commutation laws of creation and annihilation operators; in the former case the commutator of the spin operators is itself an operator, whereas in the latter case the anticommutator of the momentum operators is a c-number. In consequence the summation of the perturbation series is rather more complicated in the present case which entrains, in this respect, some problems more general than their counterparts that arise in dealing with either fermion or boson systems.

2. – Formulation of the problem.

We consider a three-dimensional lattice in the form of a cube of $N = N_0^3$ atoms, where N_0 is even, and of the type that is divisible into two sublattices. We assume cyclic boundary conditions, such that the atoms on any given face of the cube have as outside nearest neighbours the atoms on the opposite face. The two-dimensional lattice can be included here by making obvious changes.

The basic states of the system are defined in Section **1**. We define creation and annihilation operators for each site i, S_i^+ and S_i^- respectively, where

$$S_i^+ = S_i^x + iS_i^y \quad \text{and} \quad S_i^- = S_i^x - iS_i^y , \tag{2}$$

so that the Hamiltonian (1) can be written

$$H = J \sum_{\langle ij \rangle} S_i^z S_j^z + \tfrac{1}{2} J \sum_{\langle ij \rangle} (S_i^+ S_j^- + S_i^- S_j^+) . \tag{3}$$

For convenience, we now rotate the spin axes at every point on one sublattice, say B, through an angle of 180° around the S^x-axis. Thus

$$S_j^z \to -S_j^z , \quad S_j^+ \to S_j^- , \quad S_j^- \to S_j^+ \qquad \text{for all } j \text{ on } B,$$

and (3) becomes

$$H = -J \sum_{\langle ij \rangle} S_i^z S_j^z + \tfrac{1}{2} J \sum_{\langle ij \rangle} (S_i^+ S_j^+ + S_i^- S_j^-) \tag{4a}$$

$$\equiv H_0 + V , \tag{4b}$$

where H_0 is the diagonal part of H and V the off-diagonal part.

Strictly speaking, different symbols should be used in (4) for the spin-vector components at all sites j on B after the rotation. But in (4), and hence-

forth, all operators and states will be defined with respect to the axes after rotation, so there is no confusion.

Let $|\Phi_n\rangle$, $n=1, 2, ..., L\equiv(2S+1)^N$ denote the orthonormal basic states, and let $|\Phi_1\rangle$ denote the state with $S_i^z=-S$, all i, $|\Phi_L\rangle$ the state with $S_i^z=S$, all i.

If s_i' specifies the eigenvalue of S_i^z in a state $|\Phi_m\rangle$, then $|\Phi_m\rangle$ will be said to have spin $s_i=(s_i'+S)$ at i. Writing $|\Phi_m\rangle=|...\,s_i...\rangle$, specifying only the spin at i, the properties of S_i^+ and S_i^- are

$$S_i^+|...\,s_i...\rangle=(s_i+1)^{\frac{1}{2}}(2S-s_i)^{\frac{1}{2}}|...\,(s_i+1)...\rangle\,, \tag{5a}$$

$$S_i^-|...\,s_i...\rangle=s_i^{\frac{1}{2}}(2S-s_i+1)^{\frac{1}{2}}|...\,(s_i-1)...\rangle\,, \tag{5b}$$

where s_i can take values $0, 1, ..., 2S$. We say that S_i^+ and S_i^- respectively create and destroy one spin at i. In addition, the commutation rules in convenient form are

$$[S_i^+, S_j^-]=2S_i^z\delta_{ij}\,, \tag{6a}$$

$$S_i^-S_j^z=(S_j^z+\delta_{ij})S_i^-\,, \quad \text{and} \quad S_j^zS_i^+=S_i^+(S_j^z+\delta_{ij})\,, \tag{6b}$$

where δ_{ij} is the Kronecker δ.

The state $|\Phi_1\rangle$ has the properties:

$$S_i^z|\Phi_1\rangle=-S|\Phi_1\rangle\,, \quad \text{all } i\,, \tag{7a}$$

$$S_i^-|\Phi_1\rangle=0\,, \quad \text{all } i\,, \tag{7b}$$

and for any integer $l>0$,

$$(S_i^+)^{2S+l}|\Phi_1\rangle=0\,, \quad \text{all } i\,, \tag{7c}$$

using (5a) for the last property. Every other state $|\Phi_n\rangle$ can be obtained from $|\Phi_1\rangle$ by applying the requisite creation operators and multiplying by a normalizing factor; thus the state $|\Phi_m\rangle$ with spin s_i at i, etc., may be written $\mathcal{N}_m\prod_i(S_i^+)^{s_i}|\Phi_1\rangle$, where $\mathcal{N}_m$ is the normalizing factor. In addition, we define E_0 as the eigenvalue of H_0 for $|\Phi_1\rangle$, and $E_0+E_{r_1...r_i...}$, where $r_i\equiv i^{s_i}$, denotes the eigenvalue of H_0 for $|\Phi_m\rangle$. Sometimes the latter will be called the excitation energy of the state $|\Phi_m\rangle$. In particular we have

$$|\Phi_L\rangle=\mathcal{N}_L\prod_{i=1}^{N}(S_i^+)^{2S}|\Phi_1\rangle\,, \tag{8a}$$

with

$$\mathcal{N}_L=[(2S)!]^{-N}\,, \tag{8b}$$

and

$$E_{1^{2S}\ldots i^{2S}\ldots} = 0 \,. \tag{8c}$$

Finally, it is convenient to introduce the operator $R = \prod_{i=1}^{N} R_i$, where, $R_i^2 = I$, I being the identity operator, and [6]

$$R_i S_i^- R_i^{-1} = S_i^+ \,, \quad R_i S_i^+ R_i^{-1} = S_i^- \,, \quad R_i S_i^z R_i^{-1} = -S_i^z \,, \qquad \text{all } i \,,$$

and

$$R_i |\ldots s_i \ldots\rangle = |\ldots (2S - s_i) \ldots\rangle \,, \qquad \text{all } i \,.$$

R is Hermitian, and has the properties

$$R^2 = I \,, \tag{9a}$$

$$[R, H_0] = 0 \,, \qquad [R, V] = 0 \,, \tag{9b}$$

$$R|\Phi_1\rangle = |\Phi_L\rangle \,. \tag{9c}$$

We now wish to find the ground state of (4), assumed ordered, and its energy, by treating V as a perturbation on one of the ground states of H_0. From (4), (9*b*), and (9*c*), these are

$$|\Phi_\pm\rangle = 2^{-\frac{1}{2}}(|\Phi_1\rangle \pm |\Phi_L\rangle) = 2^{-\frac{1}{2}}(I \pm R)|\Phi_1\rangle \,, \tag{10}$$

with energy

$$E_0 = -\tfrac{1}{2} NZJS^2 \,, \tag{11}$$

where Z is the number of n.n. of each lattice point. Assuming the convergence of the perturbation series, there will then be, corresponding to (10), two perturbed states $|\Psi_\pm\rangle = \sum_{n=1}^{L} C_n^\pm |\Phi_n\rangle$ with energies $E_\pm = E_0 + \Delta E_\pm$ respectively and, subject to our assumptions, one will be the ground state of H. From (9*b*) the $|\Psi_\pm\rangle$ have the same symmetry properties with respect to R as $|\Phi_\pm\rangle$, respectively. It is possible to conclude which has the lower energy

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[6] R_i does not correspond to a pure space rotation of the axes at i, but it is introduced only for mathematical convenience and we are not interested in its physical interpretation. In matrix representation, R_i is a matrix with ones down the skew-diagonal, and zeros elsewhere.

from a theorem of Marshall (1), which asserts that in the ground-state the coefficient of $|\Phi_m\rangle$ is of the form $C_m = (-1)^m a_m$, where a_m is real and positive, and where $m = \sum' s_i$, the primed sum being over all spins of sublattice A for $|\Phi_m\rangle$. Thus the coefficients of $|\Phi_1\rangle$ and $|\Phi_L\rangle$ are a_1 and $(-1)^{NS} a_L$ respectively, both always positive from our choice (7) of N. Therefore the symmetric state $|\Psi_+\rangle$ is the ground-state. However, it proves of interest to carry through the calculation for both states.

The interaction through V between $|\Phi_1\rangle$ and $|\Phi_L\rangle$ will be negligibly small for large systems (8), and it would make no difference to the practical evaluation of the energy if $|\Phi_1\rangle$ were taken by itself, instead of the combinations (10), as a first approximation. However, it will be seen later that the more accurate choice (10) is necessary for the rigorous derivation of the perturbation series; moreover it avoids the necessity of introducing a « staggered » anisotropy energy, which varies from site to site in the lattice, in order to split the degeneracy of $|\Phi_1\rangle$ and $|\Phi_L\rangle$, such as is used, *e.g.*, in spin-wave theory (2).

We shall apply the theorem of Gell-Mann and Low (9) to this system. The Hamiltonian (4) is replaced by the time-dependent form

$$H' = H_0 + V \exp[\alpha t], \qquad \alpha > 0,$$

so that the perturbation is regarded as being gradually switched on from time $-\infty$; the state of the system is postulated to tend to one or the other of $\exp[-iE_0 t]\,|\Phi_\pm\rangle$ as $t \to \infty$. If at time t the state is $|\Psi^\alpha_\pm(t)\rangle$, then

$$|\Psi^\alpha_\pm(t)\rangle = \exp[-iE_0 t]\, U^\alpha_t\, |\Phi_\pm\rangle$$

defines the operator U^α_t, with the boundary condition $U^\alpha_{-\infty} = I$. The time-dependent Schrödinger equation to be solved is therefore

$$(12)\quad (H_0 + V\exp[\alpha t])\exp[-iE_0 t]\, U^\alpha_t |\Phi_\pm\rangle = i\frac{\partial}{\partial t}\left(\exp[-iE_0 t]\, U^\alpha_t |\Phi_\pm\rangle\right);$$

in integral form (12) becomes

$$(13)\quad U^\alpha_t |\Phi_\pm\rangle = -i\exp[-i(H_0 - E_0)t]\int_{-\infty}^{t} dt' \exp[i(H_0 - E_0)t']\exp[\alpha t'] V U^\alpha_{t'} |\Phi_\pm\rangle .$$

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(7) This would not always be true for a linear chain with cyclic boundary conditions.

(8) P. Anderson: *Phys. Rev.*, **86**, 694 (1952).

(9) M. Gell-Mann and F. Low: *Phys. Rev.*, **84**, 350 (1951).

The well-known iterated solution of (13) is

$$U_t^{\alpha}|\Phi_{\pm}\rangle = \sum_{n=0}^{\infty} \exp[n\alpha t][E_0 - H_0 + ni\alpha]^{-1}V[E_0 - H_0 + (n-1)i\alpha]^{-1}\cdot \cdot V \ldots [E_0 - H_0 + i\alpha]^{-1}V|\Phi_{\pm}\rangle , \tag{14}$$

where the term for $n = 0$ is $|\Phi_{\pm}\rangle$.

The Gell-Mann–Low theorem then states that, for any finite t,

$$|\Psi_{\pm}\rangle = a \lim_{\alpha \to 0} \frac{U_t^{\alpha}|\Phi_{\pm}\rangle}{\langle \Phi_{\pm}|U_t^{\alpha}|\Phi_{\pm}\rangle} , \tag{15}$$

where a is a constant, and

$$\Delta E = \lim_{\alpha \to 0} i \frac{\partial}{\partial t} \ln \langle \Phi_{\pm}|U_t^{\alpha}|\Phi_{\pm}\rangle , \tag{16}$$

provided the limits exit. The time dependence vanishes on taking the limit $\alpha \to 0$.

In the sequel it will be shown that (14) can be expressed in the form:

$$U_t^{\alpha}|\Phi_{\pm}\rangle = 2^{-\frac{1}{2}}(I \pm R) \exp[U_t^{\alpha 0}(\pm) + U_t^{\alpha 1}(\pm)]|\Phi_{\pm}\rangle , \tag{17}$$

where $U_t^{\alpha 0}(\pm)$ and $U_t^{\alpha 1}(\pm)$ are (in general) infinite series, the former being a c-number and the latter containing S^+ operators, and that

$$\langle \Phi_1|\exp[U_t^{\alpha 1}(\pm)]|\Phi_1\rangle = 1 , \tag{18a}$$

$$\langle \Phi_L|\exp[U_t^{\alpha 1}(\pm)]|\Phi_1\rangle = 0 . \tag{18b}$$

Substituting (17) into (15) and (16), and using (9a) and the hermiticity of R, we get

$$|\Psi_{\pm}\rangle = 2^{-\frac{1}{2}}a(I \pm R) \lim_{\alpha \to 0} \exp[U_t^{\alpha 1}(\pm)]|\Phi_1\rangle \tag{19}$$

and

$$\Delta E_{\pm} = i \lim_{\alpha \to 0} \frac{\partial}{\partial t} U_t^{\alpha 0}(\pm) , \tag{20}$$

with the aid of (18).

We shall prove that the limits in (19) and (20) exist. We shall also show that, for large systems, $\Delta E_{\pm}$ in (20) is proportional to N, and that the difference between ΔE_{+} and ΔE_{-} is negligible—both on the supposition, of course, that the perturbation series converge.

3. – Effect of double degeneracy.

The individual terms of V in (4), *i.e.* the pairs of creation or destruction operators for spins on n.n. sites, we call interactions, and when we generate $|\Phi_L\rangle$ from $|\Phi_1\rangle$, or vice-versa, by successive interactions (not necessarily all creations), we say we reverse the state. If now we substitute V from (4) into (14), and in each term operate out the H_0's in the denominator, we find that there is a factor $1/\alpha$, singular as $\alpha \to 0$, at those positions in the term (if such exist) where the effect of the successive interactions is to generate either of the states $|\Phi_1\rangle$ or $|\Phi_L\rangle$. We can distinguish two processes in these terms. The first is the « vacuum-to-vacuum » process, which occurs in a term whenever successive interactions take us from either $|\Phi_1\rangle$ or $|\Phi_L\rangle$ back to the same state, respectively, without passing through the other at any intermediate stage. The second is the « reversal » process, and we say that a single reversal occurs whenever successive interactions take us from $|\Phi_1\rangle$ to $|\Phi_L\rangle$, or vice versa, such that nowhere in between do they take us from $|\Phi_L\rangle$ to $|\Phi_1\rangle$, or vice versa, respectively. Going from right to left, each term passes through a definite number, say η, of single reversals, where $\eta \geqslant 0$. Terms with η reversals for $\eta > 0$ are, from (8*a*) and the form of the interactions, at least of order ηSN, where the order of a term is just the number of interactions in it. Therefore, of course, for large systems they are unimportant in the numerical evaluation of the perturbation series when the latter converge rapidly enough to be of use. We must deal with the singularities arising from these reversals, however, and we shall in this section put (14) in a form convenient for taking account of them explicitly.

Since, from (9*b*) and (9*c*), R commutes with U_t^α, we have, from (14) and (10)

$$U_t^\alpha |\Phi_\pm\rangle = 2^{-\frac{1}{2}}(I \pm R) U_t^\alpha |\Phi_1\rangle \,. \tag{21}$$

Writing now $Z_l = (E_0 - H_0 + il\alpha)^{-1}$ for convenience, we define

$$U(p, q) \equiv \exp\left[(p-q)\alpha t\right] Z_p V Z_{p-1} V \ldots Z_{q+1} V \,, \quad \text{with} \quad U(q, q) = I,$$

so that

$$U_t^\alpha |\Phi_\pm\rangle = 2^{-\frac{1}{2}}(I \pm R) \sum_{p=0}^{\infty} U(p, 0) |\Phi_1\rangle \,. \tag{22}$$

We introduce the projection operator P for $|\Phi_L\rangle$:

$$P \equiv I - |\Phi_L\rangle\langle\Phi_L| = I - R|\Phi_1\rangle\langle\Phi_L| \tag{23}$$

using (9*c*) for the last equality. We further define

$$W(p, q) \equiv \exp[(p-q)\alpha t] Z_p(PV) \dots Z_{q+1}(PV), \qquad \text{with } W(q, q) = I, \tag{24}$$

(where only the combination PV appears), which is just $U(p, q)$ with $|\Phi_L\rangle$ projected out at every stage, and

$$T(p, q) \equiv \exp[(p-q)\alpha t] Z_p V[Z_{p-1}(PV) \dots Z_{q+1}(PV)], \tag{25}$$

(where only the combination PV appears inside the square bracket), representing $U(p, q)$ with $|\Phi_L\rangle$ projected out at every stage except the last. We note that the T's have the property

$$\langle \Phi_L | T(p, q) | \Phi_1 \rangle = 0 \qquad \text{for } (p-q) < SN, \tag{26}$$

since it requires at least SN interactions to generate $|\Phi_L\rangle$ from $|\Phi_1\rangle$.

We now wish to express (22) solely in terms of the W's and T's, which is most conveniently done in the following way. Consider $U(p, q)|\Phi_1\rangle$. If (as we may from (23)), we make the replacement $V \equiv [P+R|\Phi_1\rangle\langle\Phi_L|]V$ for every V in this quantity, we get, using (24), (25), and (26) the recurrence relation

$$U(p, q)|\Phi_1\rangle = W(p, q)|\Phi_1\rangle \tag{27a}$$

for $(p-q) < SN$, and

$$U(p, q)|\Phi_1\rangle = W(p, q)|\Phi_1\rangle + \sum_{r=q+SN}^{p} U(p, r) R |\Phi_1\rangle \langle \Phi_L | T(r, q) | \Phi_1 \rangle = \tag{27b}$$

$$= W(p, q)|\Phi_1\rangle + R \sum_{r=q+SN}^{p} U(p, r)|\Phi_1\rangle \langle \Phi_L | T(r, q) | \Phi_1 \rangle ,$$

for $(p-q) \geqslant SN$, where the last equality follows because, from (9*b*) and (9*c*), R commutes with all $U(p, r)$. Starting with $q = 0$ and using (27) repeatedly we develop a series for $U(p, 0)|\Phi_1\rangle$ with η terms, where $(\eta+1)SN \geqslant p > \eta SN$, and which involves only W's and T's. Substituting these series in (22) we get

$$U_t^{\alpha}|\Phi_{\pm}\rangle = 2^{-\frac{1}{2}}(I \pm R) \sum_{\eta=0}^{\infty} R^{\eta} Y_{\eta} |\Phi_1\rangle , \tag{28a}$$

where

$$\begin{cases} Y_0 = \sum_{p=0}^{\infty} W(p, 0) , \\ Y_{\eta} = \sum_{(p)} W(p_{\eta+1}, p_{\eta}) |\Phi_1\rangle \langle \Phi_L | T(p_{\eta}, p_{\eta-1}) |\Phi_1\rangle \dots \langle \Phi_L | T(p_1, 0) |\Phi_1\rangle \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } \eta > 0. \end{cases} \tag{28b}$$

Here (p) represents the set $(p_1, ..., p_\varrho, ..., p_\eta, p_{\eta+1})$, which are summed from $0, ... (p_\varrho + SN), ... (p_{\eta-1} + SN), p_\eta$ respectively, to ∞. Finally, using (9*a*) in (28*a*), we get

$$U_t^\alpha |\Phi_\pm\rangle = 2^{-\frac{1}{2}}(I \pm R) \sum_{\eta=0}^{\infty} (\pm 1)^\eta Y_\eta |\Phi_1\rangle \,. \tag{29}$$

The terms for a given η in (29) are all those with exactly η single reversals. For there are η segments of the form $\langle \Phi_L | T(p_r, p_{r-1}) | \Phi_1 \rangle$, $r = 1, ..., \eta$, and the projection operators in (25) ensure that in each of them there is one single reversal. Likewise, the definition (24) ensures that there are no reversals in the segment $W(p_{\eta+1}, p_\eta)|\Phi_1\rangle$. (We note that in Y_η each reversal has been expressed in such a way that we pass from $|\Phi_1\rangle$ to $|\Phi_L\rangle$ from right to left, and never vice versa).

In addition we can see from (24) and (28*b*) that we have $\langle \Phi_L | Y_\eta | \Phi_1 \rangle = 0$ for all η—which proves (18*b*) immediately though we return to this point in Section 5—and therefore the series in (29) has the important property that $|\Phi_L\rangle$ has been projected out.

4. – Representation by diagrams.

We now set out to calculate (29) with the aid of a diagram technique. We change the notation slightly in (29) by putting $p_\varrho = \sum_{r=1}^{\varrho} n_r$, $\varrho = 1, ..., \eta, (\eta+1)$ in each term of Y_η for all η, and in addition writing $\langle \Phi_L | T(p_{\varrho+1}, p_\varrho) | \Phi_1 \rangle = \equiv \langle \Phi_L | T(n_\varrho) | \Phi_1 \rangle$, $\varrho = 1, ..., \eta$, and $W(p_{\eta+1}, p_\eta)|\Phi_1\rangle = W(n_{\eta+1})|\Phi_1\rangle$. Denoting the ordered set $(n_1, ..., n_\eta, n_{\eta+1})$ by (n), we write (29) as

$$U_t^\alpha |\Phi_\pm\rangle = 2^{-\frac{1}{2}}(I \pm R) \sum_{\eta=0}^{\infty} (\pm 1)^\eta \sum_{(n)} Y_\eta(n) |\Phi_1\rangle \tag{30}$$

with

$$Y_\eta(n)|\Phi_1\rangle \equiv W(n_{\eta+1})|\Phi_1\rangle \prod_{\varrho=1}^{\eta} \langle \Phi_L | T(n_\varrho) | \Phi_1 \rangle \,, \tag{31}$$

where in (30) the second sum goes over all possible sets (n) such that

$$n_\varrho \geqslant NS \,, \qquad \varrho = 1, ..., \eta, \text{ and } n_{\eta+1} \geqslant 0. \tag{32}$$

Eq. (31) is that term of $Y_\eta|\Phi_1\rangle$ with η T-segments of orders $n_1, n_2, ..., n_\eta$, respectively, counting from right to left, and with a W-segment of order $n_{\eta+1}$. We now fix our attention on (31), with a given set (n), and perform the following steps I-III.

Step I. For every V in (31) we substitute the expression given in (4) and replace the H_0's in the denominator of each resulting term by the appropriate energy eigenvalues. Using the definition (24) we get

$$W(n_{\eta+1})|\Phi_1\rangle = \sum_{\chi_{\eta+1}} \varepsilon(\chi_{\eta+1}) K(\chi_{\eta+1})|\Phi_1\rangle\,, \tag{33a}$$

and using (25) we get

$$\langle\Phi_L|T(n_\varrho)|\Phi_1\rangle = \sum_{\chi_\varrho} \varepsilon(\chi_\varrho)\langle\Phi_L|K(\chi_\varrho)|\Phi_1\rangle \qquad \text{for } \varrho=1,\ldots,\eta, \tag{33b}$$

where $K(\chi_\varrho)$ is a product of n_ϱ interactions, the label χ_ϱ distinguishing the different products occurring in the ϱ-th equation [10] ($\varrho=1,\ldots,n_\eta,\ n_{\eta-1}$) of the set (33), and where

$$\varepsilon(\chi_\varrho) = (\tfrac{1}{2}J)^{n_\varrho}\exp[n_\varrho\alpha t]\prod_{\nu=1}^{n_\varrho}[(\nu - n_{\varrho-1})i\alpha - E_\nu(\chi_\varrho)]^{-1} \quad \text{for } \varrho=1,\ldots,\eta,\eta-1, \tag{34}$$

with $n_0=0$. Here $E_\nu(\chi_\varrho)$ is the excitation energy after the ν-th interaction in $K(\chi_\varrho)|\Phi_1\rangle$ counting right to left. The $K(\chi_\varrho)$ satisfy the conditions:

a) In no $K(\chi_\varrho)$ of (33*a*) or (33*b*) can we find a position to the right of which the number of S^- for any given site exceeds the number of S^+ for the same site. This is because it follows from (5) and (7*b*) that if any product violated this condition the term in (33) containing it would vanish.

b) In no $K(\chi_\varrho)$ of (33*a*) or (33*b*) can we find a position to the right of which the number of S^+ for any given site exceeds the number of S^- for the same site by more than $2S$. Again, this is because, from (5) and (7*c*), if any product violated this condition the term in (33) containing it would vanish.

c) In no $K(\chi_{\eta+1})$ of (33*a*) can we find a position to the right of which the number of S^+ for each site i ($i=1,\ldots,N$) exceeds the number of S^- for the same site by exactly $2S$. This is the consequence of the placing of the projection operators in $W(n_{\eta+1})|\Phi_1\rangle$.

d) In each $K(\chi_\varrho)$ of (33*b*) there is one and only one position with the property described in *c*), and that is after the final interaction. This is the consequence, firstly of the $\langle\Phi_L|$ on the left, and secondly of the placing of the projection operators, in the $\langle\Phi_L|T(n_\varrho)|\Phi_1\rangle$, $\varrho=1,\ldots,\eta$.

In the ϱ-th eq. (33) ($\varrho=1,\ldots,\eta,\ \eta+1$) there is one term in the sum over

[10] We drop the labels n_ϱ on the r.h.s. of eq. (33) for convenience since the set (n) is the same throughout steps I-III.

χ_ϱ for each possible ordered product of n_ϱ interactions satisfying the appropriate conditions above. Substituting (33) into (31) gives

$$Y_\eta(n)\,|\,\Phi_1\rangle = \sum_{(\chi)} \varepsilon(\chi) K(\chi_{\eta+1})\,|\,\Phi_1\rangle \prod_{\varrho=1}^{\eta} \langle \Phi_L\,|\,K(\chi_\varrho)\,|\,\Phi_1\rangle\,, \tag{35}$$

where (χ) denotes the ordered set $(\chi_1, ..., \chi_\eta, \chi_{\eta+1})$ and the sum goes over all sets (χ), and where $\varepsilon(\chi) = \prod_\varrho \varepsilon(\chi_\varrho)$ with ϱ running from 1 to $\eta+1$. We note that $\varepsilon(\chi)$ has the following property from (8*c*) and condition *d*):

$$E_{n_\varrho}(\chi_\varrho) = 0 \qquad \text{for } \varrho = 1, ..., \eta, \tag{36}$$

since just for these values of ν the state is reversed.

Two examples of terms in (35) are:

Example A. One of the terms of $Y_0(3)\,|\,\Phi_1\rangle$ for a system with $S > \frac{1}{2}$ is

$$(\tfrac{1}{2}J)^3 \exp[3\alpha t][(3i\alpha - E_{ik})(2i\alpha - E_{i^2jk})(i\alpha - E_{ik})]^{-1} \cdot \cdot (S_i^- S_j^-)(S_i^+ S_j^+)(S_i^+ S_k^+)\,|\,\Phi_1\rangle\,. \tag{37}$$

We note that this term does not exist for $S = \frac{1}{2}$, because otherwise the operator product would violate *b*) for the site *i*.

Example B. One of the terms of $Y_1(2, 2)\,|\,\Phi_1\rangle$ for the simple system with $N = 4$, $S = \frac{1}{2}$ (where the four sites, *i*, *j*, *k*, *l* are at the corners of a square) is

$$(\tfrac{1}{2}J)^4 \exp[4\alpha t][4i\alpha(3i\alpha - E_{ij})2i\alpha(i\alpha - E_{jk})]^{-1} \cdot \cdot (S_i^- S_j^-)(S_i^+ S_j^+)\,|\,\Phi_1\rangle\,\langle \Phi_L\,|\,(S_i^+ S_l^+)(S_j^+ S_k^+)\,|\,\Phi_1\rangle\,. \tag{38}$$

We see that for this term (36) is true.

Step II. In every term of (35) we use (6) to expand each $K(\chi_\varrho)$, all $\varrho(1, ..., \eta, \eta+1)$, into a sum of new operator products, such that every new product has the S^+ collected on the left and the S^- (if any) on the right, with the S^z in between.

We now give a rule [11] for the expansion of the quantity $K(\chi_\varrho)\,|\,\Phi_1\rangle$ consequent upon the above expansion of $K(\chi_\varrho)$. Let there be $m_i^+(\chi_\varrho)$, $m_i^-(\chi_\varrho)$ operators S_i^+, S_i^-, respectively, $(i = 1, ..., N)$ in $K(\chi_\varrho)$, and for each i let us distinguish the S_i^- from one another by the notation S_{it}^- $(t = 1, ..., m_1^-(\chi_\varrho))$. Let us pair off each S_{it}^- $(i = 1, ..., N;\ t = 1, ..., m_1^-(\chi_\varrho))$ with an S^+ for the same site that lies to the right of it in $K(\chi_\varrho)$, such that no S^+ belongs to more than one pair. We say that the partners in a given such pair are « contracted » with

[11] This rule is similar to Wick's theorem: G. C. Wick: *Phys. Rev.*, **80**, 268 (1950).

each other, and each distinct way of simultaneously contracting every S^- in the above manner we call an «arrangement». If follows from condition *a*) that there is at least one arrangement in $K(\chi_\varrho)$. We distinguish the arrangements by the label γ_ϱ and denote the γ_ϱ-th arrangement of $K(\chi_\varrho)$ by $K(\chi_\varrho;\gamma_\varrho)$. We prove in Appendix A that

$$K(\chi_\varrho)|\Phi_1\rangle = \sum_{\gamma_\varrho} K(\chi_\varrho;\gamma_\varrho)|\Phi_1\rangle\,. \tag{39a}$$

summed over all arrangements γ_ϱ, where the term on the r.h.s. corresponding to $K(\chi_\varrho;\gamma_\varrho)$ is

$$K(\chi_\varrho;\gamma_\varrho)|\Phi_1\rangle = \prod_{i,\,t=1}^{N,m_i^-(\chi_\varrho)} \{2[S-c_{it}(\chi_\varrho;\gamma_\varrho)]\} \prod_{i=1}^{N} (S_i^+)^{m_i(\chi_\varrho)}|\Phi_1\rangle\,, \tag{39b}$$

Here $m_i(\chi_\varrho) = m_i^+(\chi_\varrho) - m_i^-(\chi_\varrho)$, and in the first product there is one factor for each it $(i=1,\ldots,N;\ t=1,\ldots,m_i^-(\chi_\varrho))$. The number $c_{it}(\chi_\varrho;\gamma_\varrho)$ is associated with the contracted pair S_{it}^- and its partner in the arrangement γ_ϱ, say $S_{it'}^-$. It is equal to the number of S_i^- lying to the right of $S_{it'}^-$ in $K_\varrho(\chi)$ that are, in γ_ϱ, not contracted with an S_i^- lying to the right of $S_{it'}^-$ in $K(\chi_\varrho)$.

The conditions among *a*)–*d*) that are appropriate to $K(\chi_\varrho)$ place restrictions on $m_i(\chi_\varrho)$ and the $c_{it}(\chi_\varrho;\gamma_\varrho)$ in (39*b*). The only one we need to quote here is that which follows from *d*):

$$m_i(\chi_\varrho) = 2S\,, \quad \text{all } i, \text{ for every } \chi_\varrho \text{ for } \varrho=1,\ldots,\eta. \tag{40}$$

Applying the rule (39), therefore, on performing step II, we get from (39*a*) that

$$Y_\eta(n)|\Phi_1\rangle = \sum_{(\chi)} \sum_{(\gamma)} \varepsilon(\chi) K(\chi_{\eta+1};\gamma_{\eta+1})|\Phi_1\rangle \prod_{\varrho=1}^{\eta} \langle\Phi_L|K(\chi_\varrho;\gamma_\varrho)|\Phi_1\rangle\,, \tag{41}$$

where (γ) denotes the ordered set $(\gamma_1,\ldots,\gamma_\eta,\gamma_{\eta+1})$, and the second sum goes over all sets (γ) for the respective (χ); on next substituting (39*b*) we have

$$Y_\eta(n)\ |\Phi_1\rangle = (\mathcal{N}_L)^{\eta} \sum_{(\chi),(\gamma)} \varepsilon(\chi) \prod_{i,\,t=1}^{N,m_i^-(\chi)} \{2[S-c_{it}(\chi;\gamma)]\} \prod_{i=1}^{N} (S_i^+)^{m_i(\chi_{\eta+1})}|\Phi_1\rangle\,, \tag{42}$$

where in the first product the index t now runs through all the segments and goes from 1 to $m_i^-(\chi) = \sum_{\varrho=1}^{\eta+1} m_i^-(\chi_\varrho)$, so that $c_{it}(\chi;\gamma) = c_{it}(\chi_\varrho;\gamma_\varrho)$ when it belongs to the segment ϱ." In (42) we have used (40) and the fact that

$$\langle\Phi_L|\prod_i (S_i^+)^{2S}|\Phi_1\rangle = (\mathcal{N}_L)^{-1} \qquad \text{from (8}a\text{)}.$$

We give two illustrations of the procedure in Step II:

Example A. There are two arrangements of the operator product in (37), so that from (39*a*) the application of Step II gives two terms of the form (41), of which one is

$$(43)\quad (\tfrac{1}{2}J)^3 \exp[3\alpha t][(3i\alpha - E_{ik})(2i\alpha - E_{i^2jk})(i\alpha - E_{ik})]^{-1}(\dot{S}_i^-\dot{S}_j^-)(\dot{S}_i^+\dot{S}_j^+)(\dot{S}_i^+\dot{S}_k^+)\,|\,\Phi_1\rangle\,.$$

where the arrangement is conveniently indicated by the dots, which pick out the contracted pairs. Passing to the form (42) on using (39*b*) we get

$$(44)\quad (\tfrac{1}{2}J)^3 \exp[3\alpha t][(3i\alpha - E_{ik})(2i\alpha - E_{i^2jk})(i\alpha - E_{ik})]^{-1}(2S)[2(S-1)]\,S_i^+S_k^+\,|\,\Phi_1\rangle$$

Example B. The products in (38) have only one arrangement each, so that from (39*a*) the application of Step II yields the one term

$$(45)\quad (\tfrac{1}{2}J)^4 \exp[4\alpha t][4i\alpha(3i\alpha - E_{ij})2i\alpha(i\alpha - E_{jk})]^{-1}\cdot$$

$$\cdot(\dot{S}_i^-\dot{S}_j^-)(\dot{S}_i^+\dot{S}_j^+)\,|\,\Phi_1\rangle\,\langle\Phi_L\,|\,(S_i^+S_l^+)(S_j^+S_k^+)\,|\,\Phi_1\rangle\;=$$

$$(46)\qquad = (\mathcal{N}_L)^{-1}(\tfrac{1}{2}J)^4 \exp[4\alpha t][4i\alpha(3i\alpha - E_{ij})2i\alpha(i\alpha - E_{jk})]^{-1}(2S)^2\,|\,\Phi_1\rangle$$

using (39*b*). (Though $\mathcal{N}_L = 1$ and $2S = 1$ here, we leave the symbols for clarity.)

Step III. Each term (χ), (γ) of (42) can be represented by a diagram, constructed by referring to the form (41). For this purpose the $K(\chi_\varrho;\gamma_\varrho)$ are most conveniently written with the contracted pairs picked out as in examples (43) and (45). We draw $(\eta+1)$ sets of horizontal broken lines, one set above the other, with n_ϱ lines in the ϱ-th set, counting from the lowest, and draw η horizontal full lines each separating a set from the one above. We mark a point on each broken line. Then the ϱ-th set corresponds to the ϱ-th segment in the term, and the n_ϱ points represent the interactions in $K(\chi_\varrho;\gamma_\varrho)$, where ascending order in the diagram corresponds to right-to-left order in the term. It is convenient to picture this order as measured in time, where the broken lines represent interaction times, and where successive interactions are separated by unit time with the first interaction at time 1. The full lines we call « reversal » lines since, as we see (by referring to (41)), they are drawn where the state has become completely reversed from $|\,\Phi_1\rangle$ to $|\,\Phi_L\rangle$, starting again from $|\,\Phi_1\rangle$ immediately above. For each contracted pair in $K(\chi_\varrho;\gamma_\varrho)$, $\varrho = 1,\ldots,\eta,\ \eta+1$, we draw an « internal » line joining the interaction points of the two partners, and label it with the lattice site involved. For each uncontracted S^+ in $K(\chi_\varrho;\gamma_\varrho)$, $\varrho = 1,\ldots,\eta$, we draw an appropriately labelled « bound » line from its interaction point to the reversal line immediately above,

and for each uncontracted S^+ in $K(\chi_{\eta+1}; \gamma_{\eta+1})$ we draw an appropriately labelled « free » line from its interaction point leading vertically upwards out of the diagram. We note that free lines are drawn only from the top set. We refer to these three types of lines collectively as « interaction » lines, and call the interaction points « vertices ». Fig. 1 and 2 represent (44) and (46)

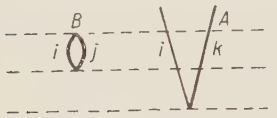

Fig. 1. – Diagram representing (44), a term of $Y_0(3)|\Phi_1\rangle$ where $S = \frac{1}{2}$, constructed by referring to (43).

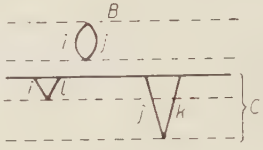

Fig. 2. – Diagram representing (46), a term of $Y_1(2, 2)|\Phi_1\rangle$ where $N = 4$, $S = \frac{1}{2}$, constructed by referring to (45).

respectively, constructed by referring to (43) and (45): in these figures the parts of the diagrams labelled A, B and C contain only free, internal, and bound lines respectively.

We combine (χ), (γ) into the single label ζ, which runs over all terms of (42) and we write the term represented by the diagram D_ζ as $G(D_\zeta)\ \Phi_1$. (42) then becomes

$$Y_\eta(n)|\Phi_1\rangle = \sum_\zeta G(D_\zeta)|\Phi_1\rangle\,. \tag{47}$$

We can obviously carry through the same Steps I-III to expand every $Y_\eta(n)|\Phi_1\rangle$ (for $\eta = 0, 1, ..., \infty$, and all (n) satisfying (32)) in the form (47). Substitution in (30) gives an expansion where each term is specified by three indices η, (n), ζ, which we combine into the label ξ running over all such sets of three; we have then

$$U_t^\alpha|\Phi_\pm\rangle = 2^{-\frac{1}{2}}(I \pm F) \sum_\xi (\pm 1)^{\eta(\xi)} G(D_\xi)|\Phi_1\,. \tag{48}$$

where $\eta(\xi)$ in the number of reversal lines in D_ξ.

We now give a prescription for obtaining the terms in (48) directly. There is one term ξ for each possible way of drawing a diagram D_ξ. All possible diagrams are obtained firstly by all different ways of drawing $(\eta+1)$ sets of $n_1, ..., n_\eta, n_{\eta+1}$, broken lines (with marked points) in accordance with (32), for each $\eta = 0, 1, ..., \infty$, in the manner described in Step III; secondly, for given η, (n), all possible ways of drawing in and labelling interaction lines under the following restrictions i)–iv) for each diagram.

i) There are two lines from each vertex, both leading either up or down, and having the labels of n.n. sites.

ii) There are precisely $2S$ bound lines of each label i $(i=1, ..., N)$ drawn to each reversal line as, *e.g.*, in Fig. 2, where $N=4$, $S=\frac{1}{2}$, we have one bound line of each of the four labels.

iii) There are not more than $2S$ free lines with the same label, nor are there $2SN$ free lines, *i.e.* $2S$ of each label 1, 2, ..., N.

iv) There are nowhere more than $2S$ interaction lines with the same label crossing simultaneously between successive interaction times, and nowhere are there $2SN$ interaction lines, *i.e.* $2S$ of each label 1, 2, ..., N crossing simultaneously between successive interaction times. Thus, *e.g.*, Fig. 1 would not be allowed when $S=\frac{1}{2}$ since the lines i would violate the first part of the restriction.

We «read off» $G(D_\xi)$ from D_ξ. Let it have $\eta(\xi)$ reversal lines, and $n_\varrho(\xi)$ broken lines in the ϱ-th set. Then

$$G(D_\xi) = (\mathcal{N}_L)^{-\eta(\xi)}(\tfrac{1}{2}J)^{\eta(\xi)} \exp[n(\xi)\alpha t]\,\varepsilon(\xi)\varkappa(\xi)\sigma(\xi)\,, \tag{49a}$$

with $n(\xi)=\sum_{\varrho=1}^{\eta+1} n_\varrho(\xi)$, and where

$$\varepsilon(\xi) = \prod_{\nu=1}^{n(\xi)} [\nu i\alpha - E_\nu(\xi)]^{-1}\,, \tag{49b}$$

$$\varkappa(\xi) = \prod_{it} 2[S - c_{it}(\xi)]\,, \tag{49c}$$

and

$$\sigma(\xi) = \prod_{i=1}^{N} (S_i^+)^{m_i(\xi)}\,. \tag{49d}$$

In (49*b*), $E_\nu(\xi)$ is the excitation energy between interaction times ν and $(\nu+1)$. $E_\nu(\xi)=0$ when $\nu=\sum_{r=1}^{\varrho} n_r(\xi)$ for $\varrho=1, ..., \eta$, (from (36)), and $E_n(\xi)$ is obtained from the free lines leading out of the diagram; otherwise $E_\nu(\xi)$ is obtained from the lines crossing between times ν and $(\nu+1)$, *e.g.* $E_2(\xi)=E_{i^2jk}$ when ξ refers to Fig. 1. $E_\nu(\xi)=0$ if no lines cross between ν and $(\nu+1)$. In (49*c*), there is one term in the product for each internal line it, where t distinguishes the internal lines i from one another. $c_{it}(\xi)$ equals the number of interaction lines of whatever type of the same label i that extend both below and above it in D_ξ (following from the interpretation of (39*b*) in terms of diagrams); *e.g.* $c_{it}(\xi)=1$ if ξ refers to Fig. 1 and it to the line i in B. Finally, in (49*d*), $m_i(\xi)$ is the number of free lines i in D_ξ.

Of the restrictions above, i) is because each interaction is either the cre-

ation or destruction of a n.n. pair, and ii)–iv) are the consequence of conditions *b*)–*d*) on the interaction products. (We note that condition *a*) is satisfied automatically by every term that appears in the sum (48).) Restriction ii) simply ensures that the reversal line represents a point at which the state is completely reversed; iii) and iv) ensure both that this occurs nowhere else, as required, and that we never have more than $2S$ spins at any site.

5. – Sum by diagrams.

Any diagram is composed of one or more separate components that cannot themselves be divided into separate parts, such A and B in Fig. 1 or B and C in Fig. 2. A component is distinguished only by the pattern and labelling of its lines, and the relative order of its vertices but not their position in the diagram. We classify the components into three types.

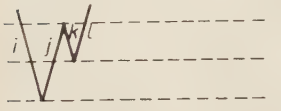

Fig. 3. – Component of type $\mathscr{A}$ (does not appear for system $N=4$, $S=\frac{1}{2}$).

Type $\mathscr{A}$, with two free lines, and any (even) number of internal lines; *e.g.* [12] Fig. 3, and A in Fig. 1. The order of components of this type is $\geqslant 1$ (where the order is the number of vertices).

Type $\mathscr{B}$, with internal lines only; *e.g.* Fig. 4 and B in Fig. 1, 2 and 5. The order of components of this type is even and $\geqslant 2$.

Type $\mathscr{C}$, constructed of $2SN$ bound lines drawn to a reversal line, and with any (even) number of internal lines; *e.g.* C in Fig. 2 and 5 where $2SN=4$. Components of this type are of even order $> NS$.

We now wish to express (48) in terms of components and their combinations. We denote any component of whatever type by F, and by $G_1(F)$

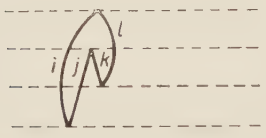

Fig. 4. – Component of type $\mathscr{B}$ (does not appear for system $N=4$, $S=\frac{1}{2}$).

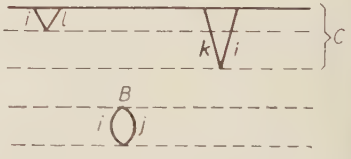

Fig. 5. – The only other non-missing combination of the components in Fig. 2.

the quantity in (48) represented by the diagram composed of only this single component; the suffix signifies here and henceforth the number of components involved. Next we let $G_\mu(F_1, F_2, \ldots, F_\mu)$ be the sum of all the quantities

[12] In Fig. 3 and all other diagrams it is implicitly assumed that lines joining at a vertex have labels belonging to n.n. sites.

in (48) represented by different diagrams composed of the specified set of components F_1, F_2, ..., $F_\mu \equiv (F)_\mu$, where no two components are identical; the diagrams in this sum differ in the « combination » of the components, *i.e.*, in the order of the vertices of each component relative to the others. For instance, $G_2(B, C)$ is a sum of two terms represented by Fig. 2 and 5 (where $N = 4$, $S = \frac{1}{2}$). As a second example, $G_2(B, B')$ for B and B' in Fig. 6 is, when $S > \frac{1}{2}$, a sum of six terms, three of which are represented by the diagrams shown and the other three by Fig. 6 with B and B' interchantged. We note here one point: in the group of diagrams composed of a given set $(F)_\mu$, some of the combinations of the components that it is possible to draw may be missing. In the first example above, all combinations of B and C are missing except the two illustrated. In the second example, on the other hand, there are no missing combinations; there would be, however, for $S = \frac{1}{2}$, since Fig. 6(*a*) and 6(*b*) are absent in this case (because otherwise the lines i violate restriction iv) of Section 4).

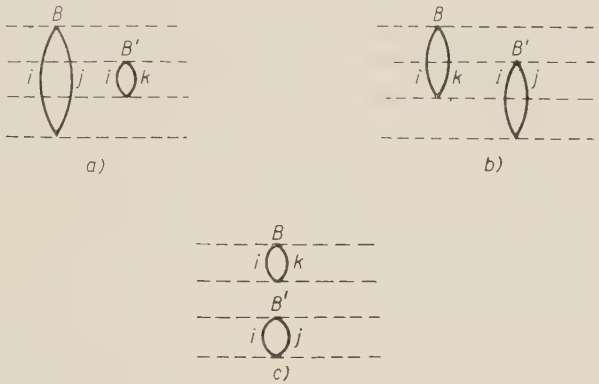

Fig. 6. – Three of the six combinations of the B and B' shown, where $S > \frac{1}{2}$: For $S = \frac{1}{2}$ (*a*) and (*b*) are missing.

Where some of the components $(F)_\mu$ are identical, we find it convenient to define $G_\mu(F_1, ..., F_\mu)$ as if permuting identical components gave different diagrams, and to remember to divide by the appropriate factor to avoid counting the same term more than once. Thus, *e.g.* if in Fig. 6 we replace k with j so that $B' = B$, the contribution in (48) from the diagrams composed of the two B's is $\frac{1}{2}G_2(B, B)$.

Noting that the number of reversal lines in a diagram is equal to the number of components of type $\mathscr{C}$, and is therefore the same, say $\eta(F)_\mu$, for each diagram composed of $(F)_\mu$, we now write (48) as

$$U_t^\alpha |\Phi_\pm\rangle = 2^{-\frac{1}{2}}(I \pm R) \sum_{\mu=0}^{\infty} (\mu!)^{-1} \sum_{(F)_\mu} (\pm 1)^{\eta(F)\mu} G_\mu(F_1, ..., F_\mu) |\Phi_1\rangle , \tag{50}$$

with $G_0 \equiv I$. Here the second sum means that each of the set $(F)_\mu$ runs independently over all possible components, and the factorial is the required factor ensuring that each term is counted once only.

All components F appearing in (50) satisfy individually, of course, the restrictions i)–iv) of Section **4**. It follows that all the combinations that it is possible to draw of any given set $(F)_\mu$ also satisfy i) and ii); but, on the other hand, they do not necessarily satisfy iii) and iv). The combinations that are missing, therefore, are those violating either of these two restrictions. There are two cases. Firstly, if any one combination of a set $(F)_\mu$ is missing because it violates iii), then all diagrams composed of $(F)_\mu$ are missing since they all have the same free lines. In this case $G_\mu(F_1, ..., F_\mu) = 0$ in (50), and we exclude such sets from further consideration until eq. (55). Secondly, therefore, the combinations of any other set that are missing are those that violate iv); this is true *e.g.* for the diagrams composed of the B and C of Fig. 2 and 4, or the B and B' of Fig. 6 when $S = \frac{1}{2}$.

Let us consider now any set $(F)_\mu$ in (50) and apply prescription (49) to $G_\mu(F_1, ..., F_\mu)$. Each diagram $D_{\xi'}$ composed of $(F)_\mu$ has the same order, say n', the same number of reversal lines, say η' and the same free lines so that $\sigma(\xi)$ is the same, say σ', for all ξ'. Thus

$$G_\mu(F_1, ..., F_\mu) = (\mathcal{N}_L)^{-\eta'}(\tfrac{1}{2}J)^{n'} \exp[n'\alpha t]\sigma' \sum_{\xi'} \varepsilon(\xi')\varkappa(\xi') \tag{51}$$

summed over all $D_{\xi'}$. Consider next, for the same set $(F)_\mu$, the product

$$\prod_{\tau=1}^{\mu} G_1(F_\tau) = (\mathcal{N}_L)^{-\eta'}(\tfrac{1}{2}J)^{n'} \exp[n'\alpha t]\sigma' \prod_{\tau=1}^{\mu} \varepsilon(\tau)\varkappa(\tau) \tag{52}$$

using (49) for each $G_1(F_\tau)$. At this point it is convenient to define two components as « overlapping » if one involves lattice sites, either in common with, or n.n. to, any of those involved in the other, and we also say in this case that the lattice regions involved by the two components « overlap »: the B of C of Fig. 5 overlap, for example, as do also the B and B' of Fig. 6. We shall now demonstrate that

$$G_\mu(F_1, ..., F_\mu) = \prod_{\tau=1}^{\mu} G_1(F_\tau) \tag{53}$$

if, and only if, $(F)_\mu$ is a set of mutually non-overlapping components. For, firstly, if the $(F)_\mu$ are non-overlapping, each of the following, A)–C), holds in (51):

A) $\varkappa(\xi') = \prod_{\tau=1}^{\mu} \varkappa(\tau)$, all ξ'. For, from (49*c*), $\varkappa(\xi')$ can be interpreted as a product of μ factors; one factor for the group of lines belonging to F_τ, for each $\tau = 1, ..., \mu$. Since by hypothesis F_τ has no label in common with any other component, the corresponding factor depends only on the disposition of the lines in F_τ (again from (49*c*)) and is hence $\varkappa(\tau)$; the result is thus proved.

B) There are no missing combinations in the sum over ξ'. For, if there were, they would be excluded by restriction iv), as we have seen above. But iv) is a condition on the disposition of lines of the same label, and since no two components have a label in common, iv) exerts no restriction on their combinations; hence none are excluded.

C) For each ξ', the $\varepsilon(\xi')$ (defined in (49*b*)) has the property that for every ν, $E_\nu(\xi') = \sum_{\tau=1}^{\mu} E_\nu(\xi', \tau)$, where $E_\nu(\xi', \tau)$ is the excitation energy found separately from the group of lines of E_τ crossing between times ν and $(\nu+1)$ (if there are none such $E_\nu(\xi', \tau) = 0$). This is because, by hypothesis, these groups ($\tau = 1, ..., \mu$) involve mutually non-overlapping lattice regions (*i.e.* further apart than n.n.) and in this case (from the form of H_0) the total excitation energy is always just the sum of the excitation energies in the separate regions.

Thus

$$\sum_{\xi'} \varepsilon(\xi')\varkappa(\xi') = \prod_{\tau=1}^{\mu} \varkappa(\tau) \sum_{\xi'} \varepsilon(\xi') = \prod_{\tau=1}^{\mu} \varkappa(\tau)\,\varepsilon(\tau)\,, \tag{54}$$

where the first step follows from *A*) and the second, following from *B*) and *C*), uses the important equality proved by GOLDSTONE ([5,13]). Hence when the $(F)_\mu$ are non-overlapping (51) and (52) are equal and (53) is true. Secondly, on the other hand, if some of $(F)_\mu$ overlap, one or more of *A*)–*C*) do not hold. For let, *e.g.* F_1 and F_2 overlap, and consider any one of those combinations where there are line(s) of F_1 and line(s) of F_2 bearing labels either in common, or that refer to n.n., crossing simultaneously between two consecutive times, say ν and $(\nu+1)$. Either this combination is missing, or (again from the form of H_0) $E_\nu(\xi') \neq \sum_{\tau=1}^{\mu} E_\nu(\xi', \tau)$, where ξ' refers to the diagram in question, since here the groups for $\tau = 1$ and 2 are overlapping. Thus either *B*) or *C*) does not hold; this is sufficient to render (54) invalid, and hence (53) is not true.

([13]) In the system studied by GOLDSTONE, the parallels to A)-C) are always automatically true.

For an example, suppose $(F)_\mu$ is just the B and B' of Fig. 6, and consider Fig. 6(a) or 6(b). If $S = \frac{1}{2}$ they are missing. Otherwise letting ξ' refer to either diagram and $\tau = 1, 2$ to B, B', respectively, we have, using (49b), that $E_2(\xi') = E_{i^2jk}$, and we have also $E_2(\xi', 1) + E_2(\xi', 2) = E_{ij} + E_{ik}$; these expressions are not equal as may easily be verified.

Quite generally we can say that when the $(F)_\mu$ overlap, then certainly B) and/or C) will not hold; this suffices to render (54) invalid and so (53) does not hold.

We can picture the components as « disturbances » in the lattice due to the perturbation in the regions corresponding to the lattice sites involved, which « interfere » with one another when the regions overlap in a way that leads to the violation of (53). To account for overlap we replace (53) by the following equations [14]:

$$G_1(F) \equiv \Lambda_1(F)\,, \qquad \text{all } F, \tag{55a}$$

$$G_2(F_1, F_2) = \Lambda_1(F_1)\Lambda_2(F_2) + \Lambda_2(F_1, F_2)\,, \qquad \text{all pairs } F_1, F_2, \tag{55b}$$

$$\begin{aligned} G_3(F_1, F_2, F_3) &= \Lambda_1(F_1)\Lambda_1(F_2)\Lambda_1(F_3) + \Lambda_1(F_1)\Lambda_2(F_2, F_3) + \\ &+ \Lambda_1(F_2)\Lambda_2(F_3, F_1) + \Lambda_1(F_3)\Lambda_2(F_1, F_2) + \Lambda_3(F_1, F_2, F_3)\,, \quad \text{all } F_1, F_2, F_3, \end{aligned} \tag{55c}$$

. . . . . .

$$G_\mu(F_1, \ldots, F_\mu) = \sum \Lambda_{\mu_1}\Lambda_{\mu_2}\ldots\Lambda_{\mu_m}\,, \quad \sum_{r=1}^{m} \mu_r = \mu\,, \qquad \text{all } (F_\mu), \tag{55\mu}$$

. . . . . .

where there is one term in the sum of (55μ) for each partition of μ, and for each different way of distributing the F among the groups of $\mu_1, \ldots, \mu_m$ for a given partition (we remember here the convention that identical F are treated as distinct from one another). There is one equation in (55) for each possible set $(F)_\mu$ for each $\mu = 1, \ldots, \infty$, and we now include the sets for which all combinations are missing; as $G_\mu(F_1, \ldots, F_\mu) = 0$ for these we put zero on the l.h.s. These equations hold with or without overlap, and the equation for $(F)_\mu$ defines $\Lambda_\mu(F_1, \ldots, F_\mu)$.

[14] These equations for the components are similar in form to Ursell's cluster equations for the imperfect gas potential as presented *e.g.* in Ter Haar: *Elements of Statistical Mechanics* (New York, 1954), chap. 8; here, overlapping components replace interacting particles.

We shall prove that the Λ's have the property

$$\Lambda_\mu(F_1, ..., F_\mu) = 0 \tag{56}$$

if at least one of $(F)_\mu$ does not overlap any of the others. This is obvious for $\Lambda_2(F_1, F_2)$ in (55*b*), for, if F_1 and F_2 do not overlap, (53) holds and $G_2(F_1, F_2) = G_1(F_1)G_1(F_2) \equiv \Lambda_1(F_1)\Lambda_1(F_2)$ so that $\Lambda_2(F_1, F_2) = 0$. The proof for $\Lambda_\mu(F_1, ..., F_\mu)$ now follows by induction. Supposing that (56) holds for all $\Lambda_\tau(F_1, ..., F_\tau)$, $\tau < \mu$, we can show that $\Lambda_\mu(F_1, ..., F_\mu) = 0$ if the $(F)_\mu$ can be divided into two (non-null) subsets $(F_1, ..., F_\varrho)$ and $(F_{\varrho+1}, ..., F_\mu)$ such that no member of one subset overlaps any member of the other. For, consider all those (non-missing) diagrams composed of two fixed combinations $\mathscr{F}_\beta$ of $(F_1, ..., F_\varrho)$ and $\mathscr{F}_\gamma$ of $(F_{\varrho+1}, ..., F_\mu)$, respectively; we regard $\mathscr{F}_\beta$ and $\mathscr{F}_\gamma$ as « compound » components so that these diagrams differ from one another only in the combination of $\mathscr{F}_\beta$ and $\mathscr{F}_\gamma$. By hypothesis $\mathscr{F}_\beta$ and $\mathscr{F}_\gamma$ do not overlap so that the sum of the terms represented by these diagrams is $G_\mu(\mathscr{F}_\beta, \mathscr{F}_\gamma) = G_\varrho(\mathscr{F}_\beta)\, G_{\mu-\varrho}(\mathscr{F}_\gamma)$ from (53). Thus

$$G_\mu(F_1, ..., F_\varrho, F_{\varrho+1}, ..., F_\mu) = \sum_\beta \sum_\gamma G_\varrho(\mathscr{F}_\beta) G_{\mu-\varrho}(\mathscr{F}_\gamma) = \tag{57}$$

$$= G_\varrho(F_1, ..., F_\varrho) G_{\mu-\varrho}(F_{\varrho+1}, ..., F_\mu)\,,$$

where in the double sum β and γ go over all $\mathscr{F}_\beta$ and $\mathscr{F}_\gamma$; *i.e.* over all (non-missing) combinations of $(F_1, ..., F_\varrho)$ and $(F_{\varrho+1}, ..., F_\mu)$ respectively. Consequently, expressing both sides of (57) in terms of the Λ's from (55), and remembering that all those terms on the l.h.s. that contain Λ's involving components from both subsets are zero, by hypothesis, with the exception of $\Lambda_\mu(F_1, ..., F_\mu)$, we are left, on cancellation, with the equation $\Lambda_\mu(F_1, ..., F_\mu) = 0$. Hence (56) is proved.

Substituting eqs. (55) into (50) we get, on rearranging terms, and writing Λ_μ for $\Lambda_\mu(F_1, ..., F_\mu)$ and η_μ for $\eta(F)_\mu$,

$$U_t^\alpha |\Phi_\pm\rangle = 2^{-\frac{1}{2}}(I \pm R) \exp\Big[\sum_{\mu=1}^{\infty} (\mu!)^{-1} \sum_{(F)\mu} (\pm 1)^{\eta_\mu} \Lambda_\mu\Big] |\Phi_1\rangle\,. \tag{58}$$

From eqs. (55), and the prescription (49), we see that $\Lambda_\mu(F_1, ..., F_\mu)$ has the same product of S^+ operators, if any, as $G_\mu(F_1, ..., F_\mu)$ (namely the σ' of (51)). This product is « read off » from the free lines in the diagrams composed of $(F)_\mu$, and these lines all belong to components of type $\mathscr{A}$. If, therefore, $(F)_\mu$ contains no components of type $\mathscr{A}$, then $\Lambda_\mu(F_1, ..., F_\mu)$ is just a *c*-number. We will denote such a set by $(F)'_\mu$, and a set with at least one component of

type $\mathscr{A}$ we denote by $(F)''_\mu$; defining $\Lambda'_\mu(F_1, ..., F_\mu)$, $\Lambda''_\mu(F_1, ..., F_\mu)$ and η'_μ, η''_μ accordingly, we write (58) as

$$(59) \quad U_t^\alpha |\Phi_\pm\rangle = 2^{-\frac{1}{2}}(I \pm R) \exp\Big[\sum_{\mu=1}^{\infty} (\mu!)^{-1} \sum_{(F)'_\mu} (\pm 1)^{\eta'_\mu} \Lambda'_\mu\Big] \cdot \exp\Big[\sum_{\mu=1}^{\infty} (\mu!)^{-1} \sum_{(F)''_\mu} (\pm 1)^{\eta''_\mu} \Lambda''_\mu\Big] |\Phi_1\rangle ,$$

where the first exponential is a number. Eq. (17) is therefore proved with

$$(60) \quad U_t^{\alpha 0}(\pm) = \sum_{\mu=1}^{\infty} (\mu!)^{-1} \sum_{(F)'_\mu} (\pm 1)^{\eta'_\mu} \Lambda'_\mu ,$$

and

$$(61) \quad U_t^{\alpha 1}(\pm) |\Phi_1\rangle = \sum_{\mu=1}^{\infty} (\mu!)^{-1} \sum_{(F)''_\mu} (\pm 1)^{\eta''_\mu} \Lambda''_\mu |\Phi_1\rangle .$$

We can now prove (18*a*) and (18*b*). Consider (61). Firstly, since by definition every term on the r.h.s. of (61) contains S^- operators, we have $\langle \Phi_1 | U_t^{\alpha 1}(\pm) | \Phi_1 \rangle = 0$, *i.e.* $\langle \Phi_1 | \exp[U_t^{\alpha 1}(\pm)] | \Phi_1 \rangle = 1$. Secondly, it follows from restriction iii) on the diagrams that $\langle \Phi_L | G_\mu(F_1, ..., F_\mu) | \Phi_1 \rangle = 0$, all $(F)_\mu$, all μ. Since this applies in particular, of course, to all $(F)''_\mu$, $\langle \Phi_L | U_t^{\alpha 1}(\pm) | \Phi_1 \rangle = 0$ (via (55)) and so $\langle \Phi_L | \exp[U_t^{\alpha 1}(\pm)] | \Phi_1 \rangle = 0$. Eqs. (18) are thus both proved.

Substitution of (60) and (61) into (19) and (20) gives us the required perturbation series for $|\Psi_\pm\rangle$ and $\Delta E_\pm$, and we need to show that the limits exist. It can in fact be proved that

$$(62a) \quad \lim_{\alpha \to 0} \frac{\partial}{\partial t} \Lambda'_\mu(F_1, ..., F_\mu) \equiv i\lambda'_\mu(F_1, ..., F_\mu) ,$$

and

$$(62b) \quad \lim_{\alpha \to 0} \Lambda''_\mu(F_1, ..., F_\mu) \equiv \lambda''_\mu(F_1, ..., F_\mu)$$

exist, and that the λ'_μ's are real. We give the proof in Appendix B. (The time dependence vanishes on taking the limit since, by definition of the Λ_μ's through (55) and (49), the time only occurs in the form of powers of $\exp[\alpha t]$.) Eqs. (19) and (20) become, from (60)–(62):

$$(63) \quad |\Psi_\pm\rangle = 2^{-\frac{1}{2}} a(I \pm R) \exp\Big[\sum_{\mu=1} (\mu!)^{-1} \sum_{(F)''_\mu}^{\infty} (\pm 1)^{\eta''_\mu} \lambda''_\mu\Big] |\Phi_1\rangle ,$$

and

$$\Delta E_{\pm} = -\sum_{\mu=1}^{\infty} (\mu!)^{-1} \sum_{(F)'_{\mu}} (\pm 1)^{\eta'_{\mu}} \lambda'_{\mu} . \tag{64}$$

Here, from (56) via (62), the λ_{μ}'s have the property

$$\lambda_{\mu}(F_1, \ldots, F_{\mu}) = 0 \tag{65}$$

if at least one of $(F)_{\mu}$ does not overlap any of the others.

We have seen that components of type $\mathscr{C}$ are of order $> NS$, so that those G_{μ}, and hence, via (55) and (62), those λ_{μ} which involve such components give terms of order NS or greater in the series (63) and (64). Thus, assuming the convergence of the series to be rapid enough, we can ignore such terms when N is large. This implies that for the energy shift in particular

$$\Delta E_{+} \approx \Delta E_{-} \approx -\sum_{\mu=1}^{n\prime} (\mu!)^{-1} \sum_{(B)_{\mu}}{}' \lambda'_{\mu} \tag{66}$$

since only components of type $\mathscr{B}$ are now involved. The primes on the sums indicate that we include only the terms up to order, say, $n \ll N$, *i.e.* each term of (66) involves few components ($\leqslant n/2$ since each component is at least of second order) each of low order ($\leqslant n$). It now follows from (65) that the r.h.s. of (66) is $O(N)$, since the number of mutually overlapping sets $(B)_{\mu}$ is $O(N)$ for each $\mu = 1, \ldots, n/2$. For we can pick one component of order $\leqslant n$, say B_1, in $O(N)$ ways, and for each B_1 the number of ways in which we can pick $\mu \leqslant n/2$ components $B_2, \ldots, B_{\mu}$, each of order $\leqslant n$, that overlap B_1 and one another, is $O(1)$. We can visualize this result roughly by imagining the set of mutually overlapping B's as representing a disturbance in the lattice in a region of a volume $\ll$ the volume of the lattice; the number of such regions we can choose, differing by lattice translations, is $O(N)$, whereas the number of different disturbances we can represent in this region is $O(1)$.

A similar argument can be applied to the perturbed state $|\Psi_{\pm}\rangle$ in (63), to show that

$$|\Psi_{\pm}\rangle \approx 2^{-\frac{1}{2}} a (I \pm R) \exp\left[\sum_{\mu=1}^{n\prime} (\mu!)^{-1} \sum_{(F)''_{\mu}}{}' \lambda''_{\mu}\right] |\Phi_1\rangle , \tag{67}$$

(where the primes on the sums indicate that we include only terms of low order so that no components of type $\mathscr{C}$ are involved), and that the number of terms appearing in the exponent is $O(N)$. The constant a is adjusted so that the state is normalized.

6. – Calculation of ground state energy.

Eqs. (63) and (64) give the exact solutions for $|\Psi_{\pm}\rangle$ and $\Delta E_{\pm}$. To do the calculations to a given order, we draw all possible diagrams up to that order, use the prescription (49), evaluate the Λ_{μ}'s from (55) and then the λ_{μ}'s from (56). We show in passing that for the very simplest system [15], where $N=2$, $S=\frac{1}{2}$, we get the correct answers. For here there is only one component possible, say C', depicted in Fig. 7, and it is of type $\mathscr{C}$. Now there is just one *distinct* diagram of n-th order (for any $n=1, \ldots \infty$), composed of C' repeated n times, but since our convention is that we get different diagrams by permuting identical components, we have $n!$ terms in $G_n(C', C', \ldots, C')$. From (49), then

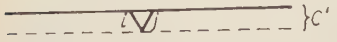

Fig. 7. – The only component for the system $N=2$, $S=\frac{1}{2}$.

$$G_n(C', \ldots, C') = n!\,(\tfrac{1}{2}J)^n \exp[n\alpha t] \prod_{m=1}^{n} (mi\alpha)^{-1}\,, \tag{68}$$

where $\mathscr{N}_L = 1$ for this system. It is simple to prove from (55) that (68) gives

$$\begin{cases} \Lambda_1(C') = \tfrac{1}{2}J \exp[\alpha t]\,(i\alpha)^{-1}\,, \\ \Lambda_n(C', \ldots, C') = 0 \qquad \text{for} \quad n \geqslant 2\,. \end{cases} \tag{69}$$

There are no Λ'_{μ}'s besides $\Lambda_1(C')$, therefore, and this can be written $\Lambda'_1(C')$, with the prime, since there is no component of type $\mathscr{A}$ involved. From (62), eq. (69) yields

$$\lambda'_1(C') = -\tfrac{1}{2}J\,, \quad \text{and} \quad \lambda'_n(C', \ldots, C') = 0 \qquad \text{for } n \geqslant 2, \tag{70}$$

and there are no λ''_{μ}. From this latter fact, and from (70), we get, respectively, that (63) and (64) are, in this case,

$$|\Psi_{\pm}\rangle = 2^{-\frac{1}{2}} a (I \pm R)\,|\Phi_1\rangle\,, \tag{71}$$

and

$$\Delta E_{\pm} = \pm\,\tfrac{1}{2}J\,, \tag{72}$$

where in (72) we have used the fact that $\eta'(C') = 1$. For this sytem $E_0 = J/4$, so we get the well-known result E_+, $E_- = 3J/4$, $-(J/4)$, respectively, where $E_{\pm} = E_0 + \Delta E_{\pm}$.

[15] We have hitherto excluded the case of a linear chain, of which $N \neq 2$ is an example. The whole analysis, however, goes through for this case, although as will be pointed out later, the series (63) and (64) do not converge for long chains.

We now consider only systems where N is large, and calculate the energy shift to fourth order for the simple plane, s.c., and b.c.c. lattices. From (66), then, putting $\Delta E = \Delta E_+ \approx \Delta E_-$, and writing ΔE_4 to denote ΔE to fourth order, we have the following expressions in which the first term contains the second-order contribution and the rest are of fourth order (no odd-order terms appear in (66) since all components of type $\mathscr{B}$ are of even order):

$$(73)\quad \left|\begin{array}{l} \Delta E_4 = -\sum_{\langle ij\rangle} \lambda_1'(B_{ij}) - \Big\{ \sum_{(ijkl)} \lambda_1'(B_{ijkl}) + \Theta\big[\sum_{\langle ijk\rangle} \lambda_1'(B_{ijk}) + \sum_{\langle jik\rangle} \lambda_1'(B_{ijk}) + \sum_{\langle ij\rangle} \lambda_1'\big) B_{ijij}\big] + \\ \quad - \sum_{\langle ijkl\rangle} \lambda_2'(B_{ij}, B_{kl}) + \sum_{\langle ijkl\rangle} \lambda_2'(B_{ij}, B_k) + \sum_{\langle ijk\rangle} \lambda_2'(B_{ij}, B_{ik}) + \frac{1}{2!} \sum_{\langle ij\rangle} \lambda_2'(B_{ij}, B_{ij}) \Big\}. \end{array}\right.$$

Here, in the sums, $\langle ij\rangle$ goes over all n.n. pairs, $\langle ijk\rangle$ over all trios where $\langle ij\rangle$ and $\langle ik\rangle$ are n.n. pairs, $(ijkl)$ over all closed chains of four sites where $\langle ij\rangle$, $\langle jk\rangle$, $\langle kl\rangle$, $\langle li\rangle$ are n.n., and $\langle ijkl\rangle$ over all open chains of four with i and l at the ends. The B_{ij} have the shape of the B or B' illustrated in Fig. 6, and the B_{ijkl}, B_{ijik}, etc., are of the shape shown in Fig. 4 (where by shape we mean the disposition of lines and vertices independently of the labels); the suffices indicate the labelling of the lines from left to right in the drawing, *e.g.* Fig. 4 shows B_{ijkl}, and B_{ijik} would be obtained from it by relabelling the lines k, l with i, k, respectively. $\Theta = 0$, 1 if $S = \frac{1}{2}$, $> \frac{1}{2}$, respectively; in the former case the components inside the square brackets in (73) do not exist, as they would violate restriction iv) of Section **4**.

For $\lambda_1'(B)$ any term in the first five sums in (73) we have $\lambda_1'(B) = -i \lim_{\alpha\to 0} (\partial/\partial t) G_1(B)$ from (62*a*) and (55*a*), and for $\lambda_2'(B, B')$ any term in the last five sums in (73) we have $\lambda_2'(B, B') = -i \lim_{\alpha\to 0} (\partial/\partial t)[G_2(B, B') - G_1(B) G_1(B')]$. Using the prescription (49) for the G's we calculate the λ's directly. In each sum of (73), every term has the same numerical [16] value, and after some tedious algebra we get the following expressions for the second-order and fourth-order contributions to ΔE_4, E_2 and E_4, respectively:

$$(74a)\quad E_2 = -JS^2 d_1 z_{\langle ij\rangle},$$

$$(74b)\quad \left|\begin{array}{l} E_4 = -\frac{1}{4} J d_1^2 d_2 S^4 z_{(ijkl)} - \Theta J d_1^2 d_2 [\frac{1}{4} S^4 z_{\langle ijk\rangle} + \frac{1}{4} S^3 (S-1) z_{\langle jik\rangle} + \frac{1}{4} S^3 (S-1) z_{\langle ij\rangle}] - \\ \quad - J d_1^3 d_2 S^4 z_{(ijkl)} - 2J d_1^3 d_3 S^4 z_{\langle ijkl\rangle} - \left\{ \begin{array}{l} \frac{1}{2} J d_1^3 d_2 S^3 (1 + 2S - 2SZ) \\ \frac{1}{8} J (Z-1)^{-3} \end{array} \right\} z_{\langle ijk\rangle} - \\ \quad - \left\{ \begin{array}{l} \frac{1}{4} J d_1^3 d_2 S^3 (2S + 2S^2 + 2SZ - 4S^2 Z - 1) \\ \frac{1}{8} J (Z-1)^{-3} \end{array} \right\} \frac{z_{\langle ij\rangle}}{2!}. \end{array}\right.$$

[16] This would not be so if the exchange integral J were different for different n.n. pairs.

where the upper line in the curly brackets is for $S > \frac{1}{2}$, the lower for $S = \frac{1}{2}$, and where $d_1 = (2SZ-1)^{-1}$, $d_2 = (SZ-1)^{-1}$, $d_3 = (4SZ-3)^{-1}$. Here $z_{\langle ij\rangle}$, $z_{\langle ijk\rangle} = z_{\langle ijk\rangle}$, $z_{\langle ijkl\rangle}$ and $z_{\langle ijkl\rangle}$ are respectively the number of pairs $\langle ij\rangle$, trios $\langle ijk\rangle$, groups of four $(ijkl)$ and groups of four $\langle ijkl\rangle$ in the lattice. The results for $S > \frac{1}{2}$ and $S = \frac{1}{2}$ have to be stated separately in the last two terms because, in the latter case we find in the calculation of the appropriate $\lambda_2'(B, B')$ that some combinations of the components are missing.

The $z_{\langle ij\rangle}$, $z_{\langle ijk\rangle}$, $z_{\langle ijkl\rangle}$, $z_{\langle ijkl\rangle}$ have the respective values $\frac{1}{2}NZ \times [1, 3, \frac{1}{2}, 7]$ for the simple plane lattice, $\frac{1}{2}NZ \times [1, 5, 1, 21]$ for the s.c. and $\frac{1}{2}NZ \times [1, 7, \frac{9}{4}, 20]$ for the b.c.c., where Z has the appropriate value in each case; namely $Z = 4, 6, 8$ for the simple plane, s.c. and b.c.c. lattices respectively. The numerical results for the three lattices are set out in Table I for $S = \frac{1}{2}$ and Table II for $S = 1$.

TABLE I. – *Values of* $\varepsilon_m \equiv (-2E_m/NZJ)$, *where* E_m *is the m-th order contribution to the energy, for lattices* $Z = 4, 6, 8$, *and* $S = \frac{1}{2}$. *Here* $\varepsilon = \varepsilon_0 + \varepsilon_2 + \varepsilon_4$.

Z	ε_0	ε_2	ε_4	ε
4	0.2500	0.0833	0.0247	0.3580
6	0.2500	0.0500	0.0086	0.3086
8	0.2500	0.0357	0.0022	0.2879

TABLE II. – *Values of* $\varepsilon_m \equiv (-2E_m/NZJ)$, *where* E_m *is the m-th order contribution to the energy, for lattices* $Z = 4, 6, 8$, *and* $S = 1$. *Here* $\varepsilon = \varepsilon_0 + \varepsilon_2 + \varepsilon_4$.

Z	ε_0	ε_2	ε_4	ε
4	1.0000	0.1428	0.0002	1.1430
6	1.0000	0.0909	0.0006	1.0915
8	1.0000	0.0667	0.0004	1.0671

TABLE III. – *Comparison of the values of* $(-2E/NZJ)$ *obtained by different methods for* $S = \frac{1}{2}$.

Z	ANDERSON	KUBO	MARSHALL	DAVIS	This result
4	0.329	0.323_5	0.3280	0.3320	0.3580
6	0.299_5	—	0.3008	0.2999	0.3086
8	—	0.284_5	0.2893	0.2870	0.2879

The first three columns in each table give respectively the values of $-(2E_0/NZJ) \equiv \varepsilon_0$, $-(2E_2/NZJ) \equiv \varepsilon_2$, and $-(2E_4/NZJ) \equiv \varepsilon$, where E_0 is obtained from (11). The last column gives $-(2E/NZJ) = \varepsilon_4$ where $E = E_0 + E_2 + E_4$. In Table III the values for ε obtained here are compared with those obtained

by ANDERSON ([17]), KUBO ([18]), MARSHALL ([1]) and DAVIS ([21]) for the case $S = \frac{1}{2}$. The first of these used the spin-wave theory, the next two used variational methods, and the last (see concluding remarks) a perturbation method.

We can see from Tables I and II that the convergence (as expressed by E_4/E_2) is better for $S=1$ than for $S=\frac{1}{2}$; in fact we find that for all higher values of S it remains considerably better than in the latter case, even though it appears to fluctuate with S. We consider what happens as $S \to \infty$. If we go back to series ΔE of (66) we can show that as $S \to \infty$ each λ'_μ, $\mu > 2$, $\to$ a constant, whereas each $\lambda'_1 \to S \times$ a constant. The former, therefore, decrease in relative value and can be neglected in our calculations; this is to be expected when we recall that they arise from overlap which becomes less important (relatively) as S gets larger. We may now write $\Delta E \approx S \lim_{S\to\infty} [\sum_B \lambda'_1(B)/S]$, of which the first terms are $S \lim_{S\to\infty} [(E_2 - E_4)/S]$, and are obtained directly from (74). We do not give them here; it suffices to say that for the lattice $Z = 4$, where the convergence is the slowest, we get $E_4/E_1 \approx \frac{1}{16}$. This is good, but not to good as for $S = 1$ for this lattice (see Table II). Probably the ratio passes through a minimum for some intermediate value of S. (We must remember in all this, of course, that as $S \to \infty$, E_0 goes as S^2 and swamps the energy shift ΔE; if we neglect the latter we get $E \to E_0$ for a « classical » antiferromagnet. We choose, here, however, to measure energy from the level E_0.)

7. – Concluding remarks.

We can see from Table III that for $S = \frac{1}{2}$ this fourth-order perturbation calculation gives lower values for the energy than previous methods, with the exception of Marshall's result ([19]) for $Z = 8$. Also from Tables I and II and the last paragaph of Section 6, we can conclude that the convergence up to E_4 is reasonably good in all cases except possibly for $Z = 4$, $S = \frac{1}{2}$, where $E_4/E_2 \approx \frac{1}{4}$. We may remark that in no case is the result below the lower bound $E = -\frac{1}{2}NZJ(1+1/ZS)$ given by ANDERSON ([20]). The convergence improves as Z increases, as we expect, since the spins « lock » more firmly. On the other hand we find that if we carry through the calculation for the linear chain, where $Z = 2$ (which gives us eq. (74) with $z_{(iikl)} = 0$), then—except in special

([17]) P. ANDERSON: *Phys. Rev.*, **86**, 694 (1952).

([18]) R. KUBO: *Rev. Mod. Phys.*, **25**, 344 (1953).

([19]) MARSHALL used the Bethe-Peierls method with his variational approach and so his result is not an upper bound.

([20]) P. ANDERSON: *Phys. Rev.*, **83**, 1260 (1951).

cases such as for $N = 2$ in Section 5 where the number of terms in the exact perturbation series (64) is finite—the method breaks down for small S: for example for $S = \frac{1}{2}$ we have $E_1 = NJ/4$ and $E_4 = 5NJ/16$. Again, this is not surprising since the disordering effect of the off-diagonal terms in H are greater in this case than elsewhere. This leads in fact to the crucial question as to whether this disordering effect becomes important in terms above fourth-order for the lattices with $Z \geqslant 4$. It is certainly possible that the decrease in the individual value of higher-order terms is outweighed by their rapidly increasing number, so that the series eventually diverges after passing through a minimum. The more closely packed the lattice though, the higher is the order at which such a « catastrophe » would occur and indeed the greater is the likelihood that it does not occur at all. The question is, however, unresolved: until it is answered we cannot know whether our calculation gives the ground-state, though we can say that it gives a state at a « local » energy minimum, stable against small perturbations. If the series does converge then this implies, of course, that there is order in the ground-state and we can calculate the values of the long- and short-range order for the $\Psi_\pm$ we have obtained. But because any conclusion about the order in this context rests on the convergence assumption, it would not seem to be very profitable to go through with the determination of these quantities here.

The method used in this paper can be adapted in a straightforward way to several cases where the Hamiltonian differs slightly from (1). As a first example, we can consider a lattice where H is anisotropic: $H = J \sum_{\langle ij \rangle} S_i^z S_j^z + \nu J \sum_{\langle ij \rangle} (S_i^x S_j^x + S_i^y S_j^y)$, with $0 \leqslant \nu \leqslant 1$. We need only to introduce a factor ν^n in the n-th order terms in the perturbation series, thus *e.g.* in Tables I and II we have to multiply the second and third columns by ν^2 and ν^4 respectively. For any $\nu < 1$, the convergence will evidently be improved, and indeed for ν small enough the series for the linear chain will converge. Secondly, in the case where the exchange integral is not the same for each n.n. pair so that J must be written J_{ij}, say, for $\langle ij \rangle$, the formulae are applicable up to and including (73) for ΔE_4. After that the difference lies only in the numerical evaluation of the terms. The method is also adaptable in principle to any other case where H can be split into two parts, a diagonal part H_0 with the doubly-degenerate ground-state $2^{-\frac{1}{2}}(|\Phi_1\rangle + |\Phi_2\rangle)$, and an off-diagonal part V such that $[R, H_0] = [R, V] = 0$. Eqs. (63)–(66) for the perturbation series are still formally valid; the difference lies only in the structure of the components —while the division into types still holds good. One example is a Hamiltonian including next n.n. ferromagnetic interaction. (Of course if the ground state of H_0 is non-degenerate, the method is easily reformulated in a simple way; the results are again given formally by (63) and (64) with $R \equiv 0$ and where there are no components of type $\mathscr{C}$. Eq. (66) remains valid as it is.

Very recently, a paper by DAVIS ([21]) has appeared which also attacks the same antiferromagnetic ground-state problem. Davis' method differs, however in that the Hamiltonian is cast into such a form that the Goldstone formalism can be applied directly. This leads in addition to a slightly different division into perturbed and unperturbed parts, where the degeneracy of the unperturbed ground-state disappears, though the unperturbed energy remains unchanged. The writer feels, however, that the method presented in this paper is of value, both for comparison with Davis' results and for the intrinsic interest of the procedure. In particular, the techniques for dealing with the double degeneracy of the unperturbed ground-state, and for dealing with the N-body divergences in this case where, as opposed to comparable boson or fermion systems, the separate components in a diagram overlap, may be useful for tackling similar problems in other fields.

* * *

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APPENDIX A

We show that if K is any one of the operator products $K(\chi_\varrho)$ defined in Sec. **4**, with m_i^+ operators S_i^+, m_i^- operators S_i^-, $(i = 1, \ldots N)$, then

$$K|\Phi_1\rangle = \sum_\gamma K(\gamma)|\Phi_1\rangle\,, \tag{A-1}$$

where

$$K(\gamma)|\Phi_1\rangle = \prod_{i,t=1}^{N,m_i^+} \{2[S - c_{it}(\gamma)]\} \prod_{i=1}^{N} (S_i^+)^{m_i}|\Phi_1\rangle\,, \tag{A-2}$$

with $m_i = m_i^+ - m_i^-$. The c-number $c_{it}(\gamma)$ is associated with S_{it}^- and its contracted partner $S_{it'}^+$ in the arrangement γ, and is equal to the number of S_i^+

([21]) H. L. DAVIS: *Phys. Rev.*, **120**, 789 (1960).

to the right of S_{it}^{+} that are, in ν, not contracted with an S_i^- lying to the right of S_{it}^- in K.

We write

$$K|\Phi_1\rangle = \prod_{i=1}^{N} K_i|\Phi_1\rangle \,, \tag{A-3}$$

where on the r.h.s. the operators have been rearranged so that K_i contains all those with the label i in the same relative order as in K; this process is easily done since from (6a) operators referring to different sites commute. We consider now $K_i|\Phi_1\rangle$, and label the S_i^- and S_i^+ as S_{it}^-, $t=1,\ldots,m_i^-$, and S_{ir}^+, $r=1,\ldots,m_i^+$, counting t and r from the right. Let us suppose that there are p of S_i^+ to the right of S_{i1}^- ($p \geqslant 1$ from condition (a) of Sec. **4** on $K|\Phi_1\rangle$) so that

$$K_i|\Phi_1 = K_i' S_{i1}^- S_{ip}^- \ldots S_{i1}^- \Phi_1 \,, \tag{A-4}$$

where K_i' is the part of K_i to the left of S_{i1}^-. We now use eq. (6a) p times to filter S_{i1}^- to the right, getting

$$K_i|\Phi_1\rangle = K_i' \sum_{q=1}^{p} S_{i(p+1)}^+ \ldots S_{i(q+1)}^+(-2S_i^z)S_{i(q-1)}^+ \ldots S_{i0}^+|\Phi_1\rangle + \\ + K_i' S_{ip}^+ \ldots S_{i2}^+ S_{i1}^-|\Phi_1\rangle \,, \tag{A-5}$$

where $S_{i0}^- \equiv I$, and also we put $S_{i(p+1)}^+ \equiv I$, where it occurs in the summation. The last term in (A-5) is zero by (7b). Next, using (6b) repeatedly in each of the remaining terms to filter the S_i^z to the right we have

$$K_i|\Phi_1\rangle = K_i' \sum_{q=1}^{p} S_{i(p+1)}^+ \ldots S_{i(q+1)}^+ S_{i(q-1)}^+ S_{i0}^+ [-2(S_i^z + q - 1)]|\Phi_1\rangle = \\ = K_i' \sum_{q=1}^{p} 2(S - q + 1) S_{i(p+1)}^+ \ldots S_{i(q+1)}^+ S_{i(q-1)}^+ \ldots S_{i0}^+|\Phi_1\rangle \,, \tag{A-6}$$

from (7a). We note that the q-th term in (A-6) is just that where S_{i1}^- is contracted with S_{iq}^+; it contains the factor $2[S-(q-1)]$ where $(q-1)$ is the number of S_i^+ lying to the right of S_{iq}^+ (none of which, obviously, are contracted with an S_i^- to the right of S_{i1}^-).

We now repeat the procedure for each S_{it}^-, $t=2,\ldots,m_i^+$, successively in each term of (A-6); there will always be some S_i^- lying to the right from condition a) of Sec. **4**. In this way, we see that at each stage the only S_i^+ to the right of the considered S_i^- are those not contracted at the previous stages, *i.e.* not contracted with an S_i^- to the right of the one considered. Finally, therefore, $K_i \Phi_1$ is expanded into a sum, one term for each way of contracting all the S_i^-; using γ_i to distinguish among those, we have

$$K_i|\Phi_1\rangle = \sum_{\gamma_i} \prod_{t=1}^{m_i^+} 2[S - c_{it}(\gamma_i)](S_i^+)^{m_i}|\Phi_1\rangle \,, \tag{A-7}$$

where $m_i = m_i^+ - m_i^-$, the excess of S_i^+ over S_i^- left after the S_i^- have been exhausted, and $c_{il}(\gamma_i)$ is the number of S_i^+ to the right of $S_{il'}^+$, the contracted partner of S_{il}^- in γ_i, that are not contracted with an S_i^- to the right of S_{il}^-.

From (A-3) and (A-7), the proof of (A-1) and (A-2) follows immediately.

Appendix B

We prove that

$$\lambda'_\mu(F_1, \ldots, F_\mu) = -i \lim_{\alpha \to 0} \frac{\partial}{\partial t} \Lambda'_\mu(F_1, \ldots, F_\mu)\,, \tag{B-1}$$

and

$$\lambda''_\mu(F_1, \ldots, F_\mu) = \lim_{\alpha \to 0} \Lambda''_\mu(F_1, \ldots, F_\mu)\,, \tag{B-2}$$

exist, and that the λ'_μ's are real.

We note first that any [22] set $(F)_\mu$, of either sort $(F)'_\mu$ or $(F)''_\mu$, we have, expressing the Λ_μ in terms of the G_μ's, the inverse of eq. (49):

$$\Lambda_\mu(F_1, \ldots, F_\mu) = \sum_{P_m} (-1)^{m-1}(m-1)!\; G_{\mu_1} G_{\mu_2} \ldots G_{\mu_m}\,, \quad \text{with} \quad \sum_{r=1}^{m} \mu_r = \mu\,, \tag{B-3}$$

where there is one term in the sum over P_m for each partition of μ, and for each different way of distributing the F among the subsets $\mu_1, \ldots, \mu_m$ of a given partition (identical F being regarded as distinct). m stands for the number of subsets in the partition.

We next consider $G_\mu(F_1, \ldots, F_\mu)$, writting it G_μ for short, which is by definition the sum of contributions from each non-missing diagram composed of $(F)_\mu$. Let D_ζ be any combination of $(F)_\mu$ that it is possible to draw, missing or not, and let $G_\mu|_\zeta$ denote the contribution to G_μ from D_ζ; if D_ζ violates restrictions iii) or iv) of Sec. **4** (*i.e.* previously called missing) then we put $G_\mu|_\zeta = 0$, otherwise it is written down from prescription (49). We have therefore

$$G_\mu = \sum_\zeta G_\mu|_\zeta\,, \tag{B-4}$$

where the sum over ζ here and henceforth, unless otherwise stated, goes over all possible [23] combinations of $(F)_\mu$ that can be drawn. For example there

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[22] We remember that we do not here exclude sets for which all combinations are missing and where $G_\mu\ (F_1, \ldots, F_\mu) = 0$.

[23] In the case where $(F)_\mu$ includes some components of type $\mathscr{C}$, therefore, in many combinations (previously missing) there will be components that have vertices

are 6 possible combinations of the B and B' of Fig. 6; here

$$G_2(B, B') = \sum_{\zeta=1}^{6} G_2(B, B')_\zeta ,$$

where, labelling Figs. 6 (a), (b), (c) by $\zeta = 1, 2, 3$, respectively, we have

(B-5a)
$$\begin{cases} G_2(B, B')|_1 = J^4 \exp[4\alpha t] S^3(S-1) \cdot \\ \qquad \cdot [4i\alpha(3i\alpha - E_{ij})(2i\alpha - E_{i_3 jk})(i\alpha - E_{ij})]^{-1} , & \text{for } S > \tfrac{1}{2} , \\ \qquad = 0 , & \text{for } S = \tfrac{1}{2} , \end{cases}$$

(B-5b)
$$\begin{cases} G_2(B, B')|_2 = J^4 \exp[4\alpha t] S^4 [4i\alpha(3i\alpha - E_{ik})(2i\alpha - E_{i_3 jk})(i\alpha - E_{ij})]^{-1} , \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } S > \tfrac{1}{2} , \\ \qquad = 0, \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } S = \tfrac{1}{2} , \end{cases}$$

(B-5c) $G_2(B, B')|_3 = J^4 \exp[4\alpha t] S^4 [4i\alpha(3i\alpha - E_{ik}) 2i\alpha(i\alpha - E_{ij})]^{-1}$, for all S.

We now take any one of the expressions $G_{\mu_1} \ldots G_{\mu m}$ of the r.h.s. of (B-3) where $(F)_\mu$ is divided into m subsets with μ_r in G_{μ_r} and define $G_{\mu_1} \ldots G_{\mu_m \zeta}$ as to be inferred from D_ζ in a way similar to $G_{\mu\,\zeta}$ but differing from the latter in that the F's of each subsed are to be *treated* as non-overlapping the F's of all the others. Thus, for instance, $G_{\mu_1} \ldots G_{\mu_m}|_\zeta$ is not necessarily zero when $G_{\mu}|_\zeta$ is, since in the former any label i appearing in an F of one subset is treated as referring to a different site than an i in any other, so that the consequences of restriction iii) and iv) may be different. The value of $G_{\mu_1} \ldots G_{\mu_m}|_\zeta$ (when non-zero) is written down according to prescription (49), with the modifications following the assumption of non-overlapping subsets. We find that $G_{\mu_1} \ldots G_{\mu_m\,\zeta}$ can differ from $G_{\mu\,\zeta}$ in one or more of three respects: (I) the latter may vanish while the former does not, (II) in the factor $\varepsilon(\zeta)$ (refer to (49)), and (III) in the factor $\varkappa(\zeta)$. The $\varepsilon(\zeta)$ may differ because, (refer to (49b)) for some r, the $E_r(\zeta)$ may be obtained from lines that include those from components of different subsets, involving overlapping lattice regions in the latter case, but treated by definition as non-overlapping in the former so that the excitation energy in these regions is treated as additive. The $\varkappa(\zeta)$ may differ because (refer to (49c)) in computing, say, $c_{it}(\zeta)$ for the line it we ignore all lines of the same label belonging to components not in the same subset. As an example of the above definition,

both above and below reversal lines *e.g.* the combination of B and C of Fig. 5 where B has one vertex below C and one above. In such a combination the excitation energy between the interactions just above and below the reversal line is no longer zero and is « read off » from the interaction lines crossing the reversal line.

we have, for B and B' of Fig. 6:

(B-6a) $G_1(B)G_1(B')|_1 = J^4 \exp[4\alpha t]\, S^4[4i\alpha(3i\alpha - E_{ij})(2i\alpha - E_{ij} - E_{ik})(i\alpha - E_{ij})]^{-1}$,

(B-6b) $G_1(B)G_1(B')|_2 = J^4 \exp[4\alpha t]\, S^4[4i\alpha(3i\alpha - E_{ik})(2i\alpha - E_{ij} - E_{ik})(i\alpha - E_{ij})]^{-1}$,

(B-6c) $G_1(B)G_1(B')|_3 = J^4 \exp[4\alpha t]\, S^4[4i\alpha(3i\alpha - E_{ik})\ 2i\alpha(i\alpha - E_{ij})]^{-1}$,

valid for all S. Here B and B' are in different subsets so that the i in B is treated as referring to a different site than i in B'. This has the consequence, on comparing (B-5) with (B-6), that the terms for $\zeta = 1, 2$ differ in respect (I) for $S = \frac{1}{2}$ (since they no longer vanish) and in respect (II) for all S (as $E_{i}{}^{i}{}_{ik}$ is everywhere replaced by $E_{ij} + E_{ik}$), and that the term for $\zeta = 1$ differs in respect (III) (as S replaces $(S - 1)$, *i.e.* the value of c for the line i of B changes from 1 to 0). We note that the term for $\zeta = 3$ is unchanged; we shall return to this point soon.

We can now show that

$$G_{\mu_1} \dots G_{\mu_m} = \sum_{\zeta} G_{\mu_1} \dots G_{\mu_m}|_{\zeta}\,. \tag{B-7}$$

Consider for a moment those diagrams composed of m fixed combinations $\mathscr{F}_r$, $r = 1, \dots, m$, of the m subsets; in these we regard the $\mathscr{F}_r$ as « compound » components so that they differ only in the combinations of the $\mathscr{F}_r$. The summation in (B-7) can be performed in two stages: firstly the sum over all these diagrams for a fixed set $\mathscr{F}_r$, and secondly the sum over all sets $(\mathscr{F})_m$. The first stage yields $G_{\mu_1}(\mathscr{F}_1)G_{\mu_2}(\mathscr{F}_2) \dots G_{\mu_m}(\mathscr{F}_m)$ as follows from (53) since the $\mathscr{F}_r$ are treated as a mutually non-overlapping set of components, and in the second, for each r we sum $G_{\mu_r}(\mathscr{F}_r)$ over all combinations of the r-th subset (including overlap here); hence the result follows directly. As an example of (B-7)

$$G_1(B)G_2(B') = \sum_{\zeta=1}^{6} G_1(B)G_1(B')|_{\zeta}\,,$$

as may easily be verified.

We define for each ζ

$$\Lambda_\mu F_1 \dots F_\mu)|_{\zeta} = \sum_{P_m} (-1)^{m-1}(m-1)!\, G_{\mu_1} \dots G_{\mu_m}|_{\zeta}\,, \tag{B-8}$$

and it follows immediately from (B-3), (B-7) and (B-8) that

$$\Lambda_\mu(F_1, \dots, F_\mu) = \sum_{\zeta} \Lambda_\mu(F_1, \dots, F_\mu)|_{\zeta}\,. \tag{B-9}$$

Again, as example, $\Lambda_2(B, B')|_{\zeta} = G_2(B, B')|_{\zeta} - G_1(B)G_1(B')|_{\zeta}$, all ζ, and

$$\Lambda_2(B, B') = \sum_{\zeta=1}^{6} \Lambda_2(B, B')|_{\zeta}.$$

It is convenient at this point to define two parts of a diagram as « disconnected » if they are separated either by a space where no lines cross, or by a reversal line, *e.g.* B and B' are disconnected in Fig. 6*c*), and likewise B and C in Fig. 5. We shall prove in the last paragraph that

$$\Lambda_\mu(F_1, \ldots, F_\mu)|_\zeta = 0\,, \tag{B-10}$$

if D_ζ contains any disconnected parts. We assume this for the moment, but note the close parallel with (56); we may imagine that parts of a diagram that are disconnected represent disturbances in the lattice that do not « interfere » with one another in time — since one has ceased before another begins — just as spatially distant disturbances do not « interfere » with each other. In our example, we can easily see that $\Lambda_2(B, B')_3 = 0$, since it is the difference between (B-5*c*) and (B-6*c*) which have the same value — this latter is intuitively clear in this simple case, since it can make no difference, where B and B' are disconnected, whether we treat them as overlapping or not.

From (B-8), (B-9) and (B-10), therefore,

$$\Lambda_\mu(F_1, \ldots, F_\mu) = \sum_{P_m} \sum_{\zeta}{}' (-1)^{m-1}(m-1)!\, G_{\mu_1} \ldots G_{\mu_m}|_\zeta\,, \tag{B-11}$$

where the prime indicates that the second sum goes over only those ζ with no disconnected parts. Now, take the case where $(F)_\mu$ is a set of the sort $(F)'_\mu$. For each $G_{\mu_1} \ldots G_{\mu_m}|_\zeta$ in (B-11), we see from the modified prescription (49) that, since D_ζ has no disconnected parts, no energy in the denominator of $G(\zeta)$ is zero except the last (the last is so because no lines lead out of the diagram). If n is the order of D_ζ, therefore, $E_n = 0$ and

$$\Lambda'_\mu(F_1, \ldots, F_\mu) = \frac{\exp[n\alpha t]}{ni\alpha}\, \Gamma(\alpha)\,,$$

where $\exp[\alpha t]/ni\alpha$ is a factor common to each term in (B-11), and $\lim_{\alpha \to 0} \Gamma(\alpha)$ exists, and is real since the imaginary quantities of the form $ip\alpha$ in the denominator vanish on taking the limit. Thus, using (B-1),

$$\lambda'_\mu(F_1, \ldots, F_\mu) = -\lim_{\alpha \to 0} \Gamma(\alpha)$$

exists. In the other case, where $(F)_\mu$ is a set of the sort $(F_\mu)''$, we see again from the modified prescriptions that in each $G_{\mu_1} \ldots G_{\mu_m}|_\zeta$ of (B-11) there is no zero-energy denominator including the last (since here there are always lines leading out of the diagram). It follows directly therefore that $\lim_{\alpha \to 0} \Lambda''_\mu(F_1, \ldots, F_\mu)$ exists and hence, from (B-2), so does $\lambda''_\mu(F_1, \ldots, F_\mu)$. The result is proved.

It remains to show that (B-10) is true. Consider the r.h.s. of (B-8) for a diagram D_ζ in which the $(F)_\mu$ form two groups disconnected from one another, $F_1, \ldots, F_\nu$, and $F_{\nu+1}, \ldots, F_\mu$, of $\nu > 0$ and $(\mu - \nu) = \tau > 0$ members, respec-

tively. Then each expression $G_{\mu_1} \dots G_{\mu_m}|_\zeta$ can be rewritten

$$(G_{\nu_1} G_{\tau_1})(G_{\nu_2} G_{\tau_2}) \dots (G_{\nu_m} G_{\tau_m})|_\zeta ,$$

where G_{μ_r} has been split into the product $G_{\nu_r} G_{\tau_r}$ such that G_{ν_r} and G_{τ_r} contain those F in G_{μ_r} belonging to the first and second groups respectively; if G_{μ_r} contains no F of the first group, $G_{\nu_r} = I$, and likewise for the second group. The second expression differs formally from the first in that within each of the m subsets we treat the members of the two groups as non-overlapping. But because of the separation of the two groups in D_ζ the algebraic value is the same whether we treat them as overlapping or not, *i.e.* the two expressions can differ in none of the respects (I)-(III) given above. We quote again the example $G_1(B, B')|_3 = G_1(B) G_1(B')|_3$ for Fig. 6 (*c*) where B and B' are the two disconnected « groups ». Rewriting the r.h.s. of (B-8), then, in the above manner and putting G_{ν_r} or G_{τ_r} equal to I where appropriate, as already indicated, the sum can be rewritten in the form

$$\text{(B-12)} \qquad \Lambda_\mu(F_1, \dots, F_\mu)|_\zeta = \sum d(p, q) G_{\nu_1} \dots G_{\nu_p} G_{\tau_1} \dots G_{\tau_q}|_\zeta ,$$

with $\sum_{r=1}^{p} \nu_r = \nu$, $\sum_{s=1}^{p} \tau_s = \tau$, and where the G_ν contain only F's of the first group, and the G_τ only those of the second. There is one term in (B-12) for each way of simultaneously partitioning ν and τ, and for each different way of distributing the $F_1, \dots, F_\nu$ among the p subsets $\nu_1, \dots, \nu_p$, and the $F_{\nu+1}, \dots, F_\mu$ among the q subsets $\tau_1, \dots, \tau_q$, of the given partitions, respectively. The coefficient $p(p, q)$ of $G_{\nu_1} \dots G_{\tau_q}|_\zeta$ in (B-12) is the sum of the coefficients of all those terms in (B-8) that reduce to this quantity on rewriting and has the value

$$\text{(B-13)} \qquad d(p, q) = p!q! \sum_{t=0}^{p} \frac{(-1)^{p+q-t-1}(p + q - t - 1)!}{t!(p - t)!(q - t)!} ,$$

where we have assumed $p < q$ without loss of generality. In this sum the term $t = 0$ is the coefficient of $G_{\nu_1} \dots G_{\tau_q}|_\zeta$ itself in (B-8) (which is of course unchanged on rewriting); the term for general t is the number $p!q![t!(p - q)!(q - t)!]^{-1}$ of different terms in (B-8) that have just t of the G_{μ_r} containing members of both groups, and that reduce to $G_{\nu_1}, \dots, G_{\tau_q}$, multiplied by their common coefficient $(-1)^{p-q-t-1}(p - q - t - 1)!$ Every coefficient in (B-12), therefore, is of the form (B-13), where $0 < p < q$. Now if we consider the expression $(1 - x)^p (1 - x)^{-p} = 1$ (for $0 < x < 1$), and expand the two brackets on the l.h.s. separately in powers of x, we find that for $0 < p < q$ the coefficient of x^q is just $d(p, q)$ apart from a non-zero factor; $d(p, q)$ is thus evidently zero. Hence $\Lambda_\mu(F_1, \dots, F_\mu)|_\zeta = 0$ whenever D_ζ has two disconnected parts. Thus also $\Lambda_\mu(F_1, \dots, F_\mu)|_\zeta = 0$ when D_ζ has two or more disconnected parts, since the latter is merely a special case of the former.

RIASSUNTO (*)

Ottengo serie di perturbazioni per lo stato e l'energia fondamentali dei reticoli bi- e tridimensionali con un Hamiltoniano isotropo dello spin antiferromagnetico per lo spin generale S e con condizioni limiti cicliche. Considero solo reticoli divisibili in due sottoreticoli compenetrantisi. La parte fuori la diagonale è trattata come perturbazione, e, introducendo operatori di creazione e di annientamento di spin, la serie è valutata usando una tecnica a diagrammi. Non faccio approssimazioni. Il metodo si può facilmente adottare al caso anisotropo. I due stati fondamentali degenerati non perturbati sono le combinazioni simmetriche ed antisimmetriche dei due stati completamente ordinati orientati in direzioni opposte. Ottengo una convergenza ragionevolmente buona, almeno sino al quarto ordine, per ogni reticolo considerato in tutti gli S, ed eseguo calcoli numerici dell'energia per $S=\frac{1}{2}$ o 1. I valori sono inferiori a quelli dei trattamenti precedenti in tutti i casi eccetto il b.c.c. con $S=1$. Tuttavia ho presupposto la convergenza generale della serie, ed i risultati, con le loro implicazioni di ordine nello stato fondamentale, si basano quindi su questa supposizione. Sviluppo metodi speciali nella somma dei diagrammi per trattare: (1) la doppia degenerazione dello stato fondamentale imperturbato, e (2) le apparenti divergenze ad N corpi in questo sistema in cui gli stati basilari non possono essere rappresentanti come riunioni di « particelle libere », come nel caso del bosone o del fermione, od in cui il commutatore degli operatori di annientamento e di creazione non è un numero c. Queste tecniche possono essere utili per problemi analoghi in altri campi.

(*) *Traduzione a cura della Redazione.*

Diurnal Variation and Forbush Decrease (*).

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(ricevuto il 15 Maggio 1961)

Summary. — In this work we discuss the depressions often preceding the most important Forbush decreases by less than 24 h, and certain features of a Forbush decrease such as the rarity of onsets during morning hours, the greater amplitude when they happen during the evening as well as the small increases that sometimes happen during the main decrease. We bring forward the fact that the Forbush decrease is always associated to a train of diurnal variations and vice-versa a train of diurnal variations often contains a more or less evident Forbush type decrease. Through an analysis of the energy dependence of the above pre-decreases, and of a separation of local variations from world-wide variations in a Forbush decrease, we show that the forementioned characteristics associated with a decrease, are simply due to the superposition of diurnal variations to the world-wide Forbush decrease. The train of diurnal variations, with the Forbush effect removed, has a variable length that apparently does not exceed about ten days; it is reinforced and is anticipated in phase during a period of two or three days containing the onset of the Forbush storm.

1. – Introduction.

Some time ago (1), we pointed out that at the Cosmic Ray Stations located at the middle latitudes, diurnal varions (DV) and Forbush Decreases (FD) are always associated and that a Cosmic Ray Storm (CRS), that is a group

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(**) Presently associated with the NASA, Goddard Institute for Space Studies, New York, N. Y. under a NAS-NRC Post Doctoral Research Associateship.

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of strong fluctuations of the cosmic ray intensity, with a period not greater than a day, may be interpreted as an anisotropy more or less persistent of the cosmic ray intensity in the space around the Earth.

We also pointed out that the apparent lack of onsets of Forbush decreases during the night and morning hours, which has been noted by other authors [2], has to be interpreted as a superposition of the Forbush decrease and the diurnal fluctuation.

Many authors have observed several types of variation associated with a FD as:

a) a reinforced diurnal oscillation [3,4];

b) a preliminary depression [2];

c) a world-wide increase in the intensity associated with an increase in the earth's magnetic field following a sudden storm commencement (SSC) [5];

d) a preliminary world-wide increase [6,7].

They have also observed [2] that the onset of a FD is not simultaneous at all stations but it has the tendency to anticipate in those regions, outside of the earth's magnetic field, pointing in directions between 4 and 10 h.

We believe that variations of type *a*) and *b*) belong to the same event and that the event that looks more or less as a long train of DV, is associated with the presence on the sun of a very active region.

2. – Some typical examples of CRS.

Trains of DV of this type may begin several days before or immediately before the FD.

In both cases they show to be reinforced at least during the first and the second day of the FD.

One can also observe trains of DV not associated with a FD or associated with a scarcely observable FD.

[2] A. G. Fenton, K. G. McCracken, D. C. Rose and B. G. Wilson: *Can. Journ Phys.*, **37**, 970 (1959).

[3] Y. Sekido and S. Yoshida: *Rep. Ion. Res. Japan*, **4**, 37 (1950).

[4] H. Elliot and D. W. N. Dolbear: *Journ. Atmos. Terr. Phys.*, **1**, 205 (1951).

[5] I. Kondo, K. Nagashima, S. Yoshida and M. Wada: *Proc. of the Moscow Cosmic Ray Conference*, vol. **4** (Moscow, 1960), p. 208.

[6] L. I. Dorman: *Proc. of the Moscow Cosmic Ray Conference*, vol. **4** (Moscow, 1960), p. 134.

[7] V. Sarabhai and R. Palmeira: *Nature*, **184**, 1204 (1959).

As an example we represent in Fig. 1*a*), 1*b*), 2*a*), 2*b*), some cosmic ray storms. In these figures a couple of straight lines inclined toward the right are limiting

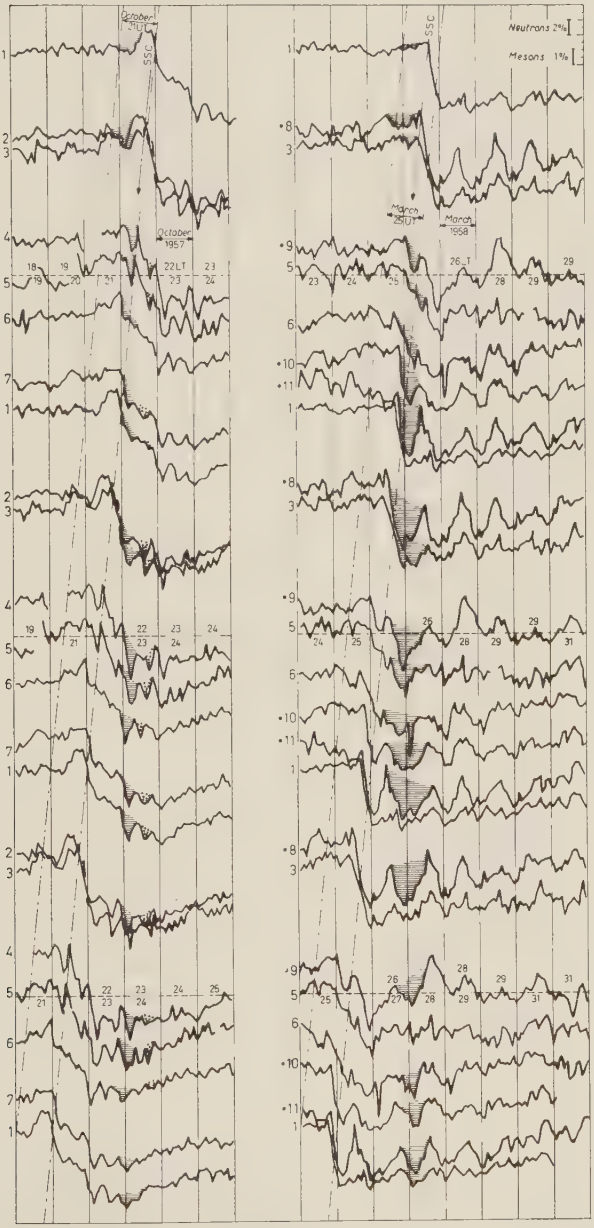

Fig. 1. – Cosmic ray storms during three earth revolutions. *a*) October 20-21-22, 1957 *b*) March 25-26-27-28, 1957. Stations, 1: Leeds + Herstmonceux. 2: Ottawa + + Mt. Washington. 3: Lincoln. 4: Mt. Wellington. 5: Yakutsk. 6: Alma Ata. 7: Uppsala + Göttingen. 8: Ottawa. 9: Hobart. 10: Moscow. 11: Weissenau. 8, 9, 10, 11: are meson recordings, the others, neutron recordings.

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portions of the intensity curve corresponding to the same universal time, whereas a couple of vertical straight lines are limiting portions of the intensity curves recorded between two successive passages of the asymptotic di-

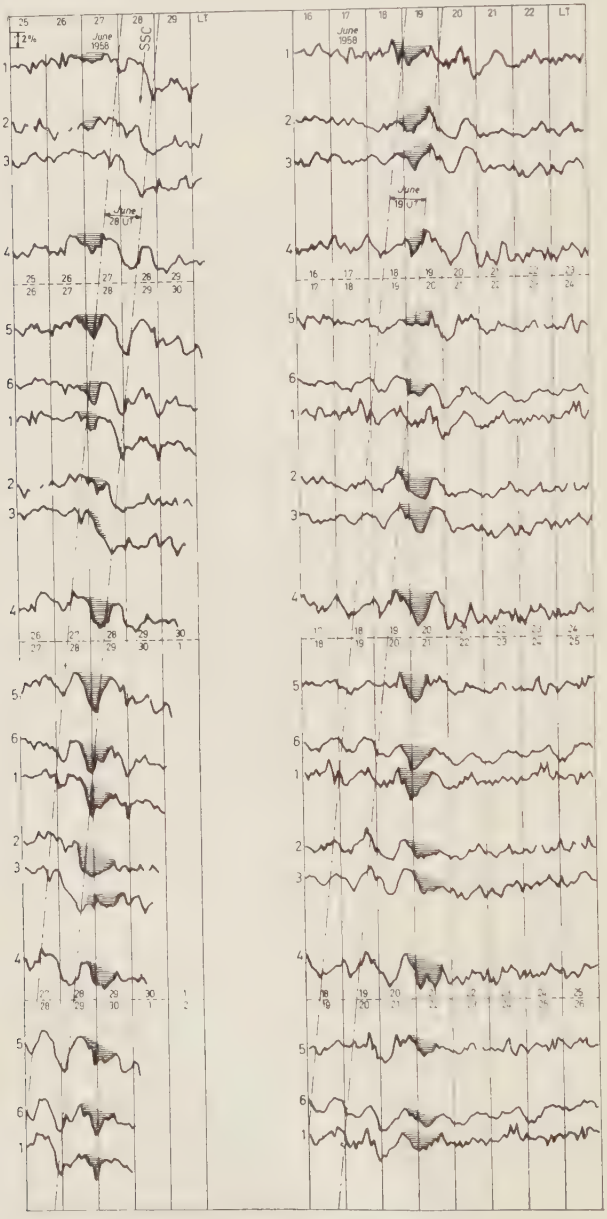


Fig. 2. – Cosmic ray storms during three earth revolutions. *a*) June 27-28-29, 1958. *b*) June 19-20-21, 1958. 1: Leeds + Herstmonceux. 2: Mt. Washington. 3: Sulphur Mt. 4: Mt. Wellington. 5: Alma Ata. 6: Zugspitze. All neutron recordings.

rections through the same meridian plane of the earth: we have represented the data of the British stations according to local time; we have then changed (an hour at most) the local time of the other stations at a different latitude in such a way that particles arriving to the British stations at a certain local time, and particles arriving to the other station at the corresponding corrected local time, have started from the same geographic longitude at infinity. In order to do this we have used the results on the asymptotic directions published by BRUNBERG ([8]).

Vertical displacements are proportional to the asymptotic longitude of the stations taken into consideration.

In the same figures we have represented four typical CRS recorded at middle latitudes, in such a way as to follow the perturbation during three terrestrial revolutions.

In Fig. 1*a*) we have represented the CRS of October 21, 1957.

We can see there that besides the FD whose onset can be associated with the SSC of 2241 UT of October 21, we can also observe anisotropies along two directions, persistent for at least two terrestrial revolutions.

In Fig. 1 *b*) we have represented the CRS of March 25, 1958 which shows a FD that can be associated with the SSC of 1540 UT superimposed to a strong DV persisting for at least four diurnal revolutions.

In the CRS of June 26, 27, 28, 1958 represented in Fig. 2 *a*), the FD which can be associated with the SSC of 1828 UT is scarcely observable, whereas in Fig. 2 *b*), in which we have represented the CRS of June 19, 20, 21, 1958, no sign of a depression can be seen.

A preliminary depression prior to FD during morning hours can be easily seen in curves marked 1, 2, 3, 4, 5 of Fig. 1 *a*); a similar pre-decrease can be observed in curves marked 8, 3, 9, 6, 10, 11, of Fig. 1 *b*).

In Fig. 1 *b*), where curves marked 8, 9, 10, 11 are meson recordings on an intensity scale doubled with respect to that one of the neutrons, we can also clearly see that, whereas the amplitude of the DV's is about the same either for mesons as for neutrons, the amplitudes of the FD's for neutrons is a little bit more than for mesons.

We can also clearly see in the same Fig. 1 that FD onsets have a tendency not to occur early in the morning.

3. – Energy dependence of pre-decrease, FD, DV.

At this point it seemed convenient to us to use the data at our disposal to do a more systematic study.

([8]) E. Å. BRUNBERG: *Tellus*, **8**, 215 (1956).

In order to do that we have considered all FD's associated to a SSC of the period August 1957-October 1958, for which neutron intensity, recorded with a standard pile at middle latitudes, within 12 hours immediately following the SSC, reaches a depression of at least 5%.

We have found 10 of such CRS (see Table I for the specification of CRS and stations from which data are taken); regarding each SSC we have then chosen recordings of six stations in such a way that each of them receives the

TABLE I.

Date	Cosmic ray stations
1) August 3, 1957	Alma Ata, Berkeley, Churchill, Leeds, Macquarie Island, Mt. Wellington, Moscow, Ottawa, Rome, Sulfur Mountain, Uppsala, Ushuaia, Weissenau, Yakutsk.
2) August 29, 1957	
3) October 21, 1957	
4) November 26, 1957	
5) February 11, 1958	
6) March 25, 1958	
7) July 21, 1958	
8) August 17, 1958	
9) August 24, 1958	
10) September 15, 1958	

SSC in one of the following four hour groups: 0-4; 4-8; 8-12; 12-16; 16-20; 20-24; in which we have divided the local day. We have taken the SSC as a temporal reference to use for a FD because it is reasonable to suppose that the solar cloud causing the SSC is in some way connected with the FD mechanism.

In order to test if the preliminary depressions are of the same nature of a FD we proceeded in the following way:

1) We subtracted the average value between the nearest maximum to the SSC and the maximum value reached 24-26 hours at most before the SSC, from the minimum value reached 12-14 hours at most before the SSC; we divided the difference by the average value of the intensity of a given component and called it amplitude of the preliminary decrease.

2) For each of the ten storms we then calculated the ratio between the amplitude of the preliminary decreases as recorded by a neutron standard counter and those recorded by a meson standard counter in a same station, using data of the stations of Weissenau, Ottawa, Hobart.

The following value was obtained:

$$\frac{\text{percentage neutron preliminary decrease}}{\text{percentage meson preliminary decrease}} = 1.33 \pm 0.10 .$$

3) Next we considered the differences between a maximum percentage value immediately near to the SSC and the minimum percentage value reached not later then twelve hours after the same SSC. This value has been called amplitude of the FD.

4) Then we proceeded as in 2) and we obtained the following value:

$$\frac{\text{neutron percentage depression}}{\text{meson percentage depression}} = 2.08 \pm 0.10 .$$

So we can conclude that the energy dependence of preliminary decreases is different from the one of the FD.

In order to see also if preliminary decreases have the same energy dependence as the DV, we have considered the trains of the DV's of the period August 1957-June 1958 (see Table II) recorded by the stations of Hobart, Sulfur Mountain, Zugspitze; with the condition that the amplitude were greater than 1.8% for neutrons and greater than 1.2% for mesons. We determined then the amplitude of the DV's and proceeded in the same way as in 2), we obtained:

$$\frac{\text{percentage amplitude of a diurnal wave for neutrons}}{\text{percentage amplitude of a diurnal wave for mesons}} = 1.31 \pm 0.04 .$$

TABLE II.

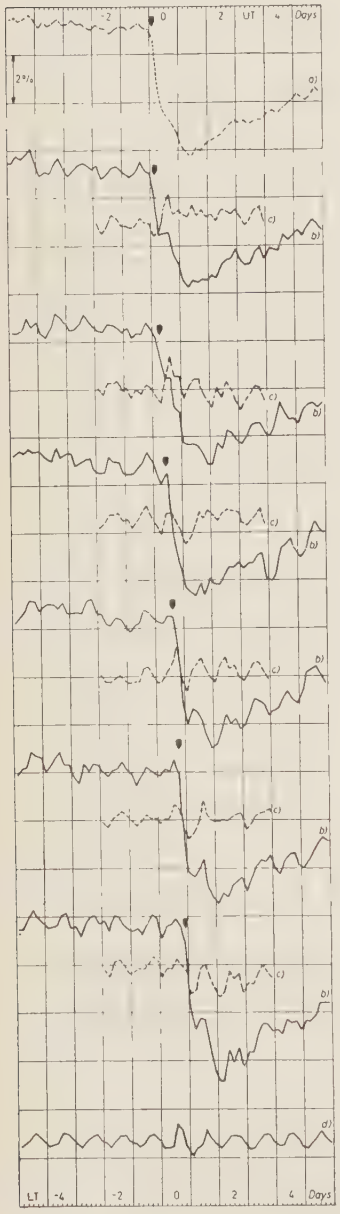
Date	Cosmic ray stations
1) August 5, 9, 10, 11, 13, 14, 1957	Hobart, Sulfur Mountain, Zugspitze.
2) September 6, 1957	
3) October 8, 9, 10, 11, 1957	
4) December 1, 2, 3, 4, 1957	
5) March 26, 27, 28, 29, 30, 1958	
6) May 21, 22, 23, 24, 25, 26, 1958	
7) June 6, 7, 8, 9, 20, 21, 27, 28, 29, 1958	

This value, which is very near to the value 1.33 ± 0.10 that had been found for preliminary decreases, together with certain small increases during the main phase of a FD (as we can see in curves marked 5, 6, 9, 10, of Fig. 1 *b*)) strongly indicates that those preliminary decreases are only the diurnal fluctuation superimposed to a FD.

4. – Separation of DV from FD in a CRS.

We then tried to separate the diurnal oscillation from the FD which we suppose superimposed in the same cosmic ray storm.

First of all we observe that if we take the average of analogous recordings of one or more storms superimposing their SSC, when these recordings have been obtained on the same earth parallel, we will obtain an average recording (world-wide recording) consisting in the average istantaneous intensity value, independent of the possible arrival directions of the particles on the earth surface around that parallel. When we take the differences between individual recordings or the average recording of a certain portion of a parallel, and the world-wide recording letting always the SSC's to coincide, we obtain the recording (residual oscillation) of the DV of that individual station or group of stations of a same portion of a parallel. Then if we take the average of the residual oscillations of each of the stations uniformly distributed over a parallel, taking care that the data of local noon of each station will be superimposed, we obtain an average behaviour of the diurnal oscillations separated from the FD.



When using neutron data of the ten storms mentioned above, and proceed as explained before, we obtain the curves of Fig. 3; the curve *a*) represents the average world-wide behaviour, curves *b*) and *c*) represent, respectively, the average behaviour and the average diurnal oscillation of those groups of stations for which the SSC happens in the following hourly intervals: 0-4; 4-8; 8-12; 12-16; 16-20; 20-24.

The curve *d*) which has been obtained taking the average of the curves *c*) and following the local day, represents the DV averaged at all groups of stations, grouped according local time, as explained before.

Fig. 3. – Separation of cosmic ray storms into world-wide and local parts: *a*) Mean worldwide behaviour of a FD. *b*) Mean behaviour of a CR storm with a FD when SSC occurs at $(0 \div 4)$, $(4 \div 8)$, $(8 \div 12)$, $(12 \div 16)$, $(16 \div 20)$, $(20 \div 24)$ h LT. *c*) Diurnal fluctuation superimposed to a FD when SSC occurs at same hours as in *b*). *d*) Mean diurnal fluctuation that has been separated from the mean world-wide behaviour of a FD.

Since the curves *b*) of Fig. 3 show a persistent DV with an amplitude which does not vary very much during the eleven days around the SSC, we wanted to see if by chance this is mainly due to only a few of the ten storms considered, whereas the majority of them has a different behaviour.

We have then divided the ten storms into two groups putting storms number 1, 3, 5, 7, 9, of Table I in the first group and in the other the remaining ones. So we have obtained curves *a*) and *b*) of Fig. 4.

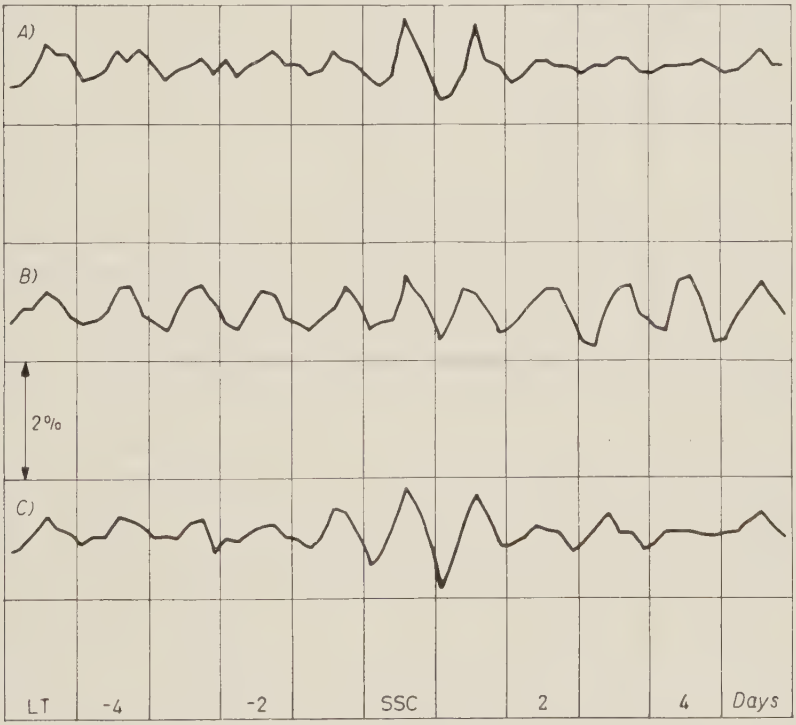


Fig. 4. – Trains of diurnal oscillations, separated from the mean worldwide behaviour of a FD.

Taking advantage of other data at our disposal we have found seven more storms (see Table III) which we have added to the first group of ten already considered: so we have obtained the curve *c*) of Fig. 4.

From an inspection of the five storms of the second group which give curve *b*) of Fig. 4, we have been able to conclude that the regular oscillation of the curve *a*) of Fig. 4 is mainly due to the storm of November 26, 1957, which shows a long train of oscillations preceeding the SSC, and to those of March 25, 1958 and August 24, 1958 which show a long train of DV after the SSC.

TABLE III.

Date	Cosmic ray stations
1) September 29, 1957 2) July 8, 1958 3) October 22, 1958 4) May 11, 1959 5) July 11, 1959 6) July 15, 1959 7) July 17, 1959	Alma Ata, Berkeley, Churchill, Leeds, Macquarie Island, Mt. Wellington, Moscow, Ottawa, Rome, Sulfur Mountain, Uppsala, Ushuaia, Weissenau, Yakutsk.

5. – Discussion of the results.

1) From a comparison between curves *b*) of Fig. 3 we see that, on the average, the preliminary decrease which when taken individually, sometimes has a considerable amplitude, now has clearly become a DV superimposed to the FD; we have to note that its average amplitude is considerably smaller with respect to the amplitude we observe sometime in singular cases. This fact may be interpreted keeping in mind their strong variability. As we have already shown by means of energy dependence, it is out of doubt that pre-decreases are of the same nature as the diurnal oscillations.

2) Looking at the curves *b*) of Fig. 3 we see that SSC's are apparently simultaneous with FD onsets only when these happen between 8 h and 20 h local time, in the other cases FD onsets appear to precede SSC's. The curves *b*) support also what found by FENTON *et al.* [2] that is the depression reaches always its maximum at stations with telescopes pointing in asymptotic directions around 90° west of the sun-earth line corresponding to the period about 0 h-6 h local time, and that the same depression is smaller for those telescopes that at the same UT are receiving from about the opposite directions.

3) From the results exposed in Fig. 3 and 4 it appears also that the amplitude of the diurnal oscillation is enhanced corresponding to the local day in which the SSC happens, as it has been already asserted by several authors. There is some indication that the diurnal amplitude is reinforced even sometime before the SSC and certainly it remains still reinforced for some more days after the SSC; the phase seems to anticipate for a few days after the SSC, as it has already been noticed by others [9].

4) There are some indications that on the average the FD onset is initially gradual and that it precedes the SSC of about 8 hours (see also in Fig. 5

[9] A. G. SANDSTRÖM: *Tellus*, **7**, 204 (1955).

the curve *a*) of Fig. 3 in bihourly scale). If for a FD onset we take the beginning of the steepest portion of the curve, then it is impossible to decide the priority order between SSC and FD.

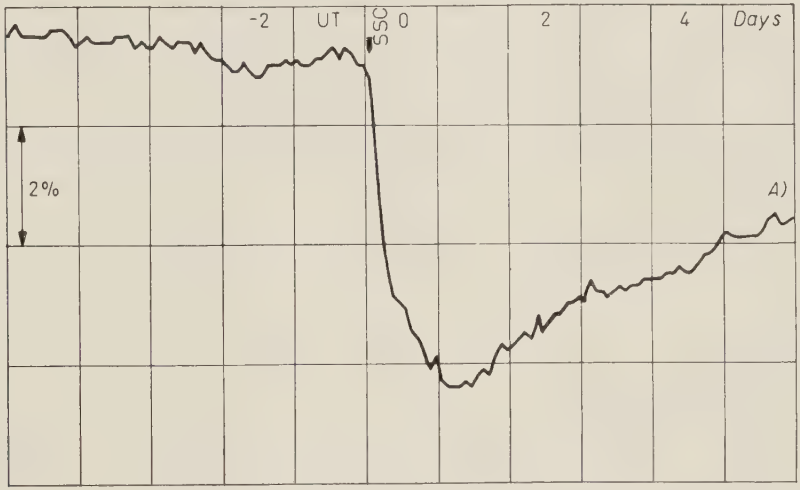


Fig. 5. – Mean world-wide behaviour of a FD separated from local variations.

The small increase immediately before the FD onset studied by Dorman seems not to have a systematic character unless the time distance before the SSC is very variable so that the average amplitude becomes very small.

On the basis of what is known, it is difficult to give a consistent interpretation of the origin of the trains of diurnal oscillations whether they are associated to FD's or not.

Following an idea of Elliot [10] one might suppose the existence of a ring current reinforced by plasma emitted by an active region of the sun; of course in this case there should be an increased magnetic field at the earth that would give a greater solar cut-off that would be the cause of a DV that sometime should be much greater than the one calculated by Elliot. Actually it has been found an interplanetary field [11] that might be due to Elliot's current system; but if that is the cause of our DV trains we should discard the idea that such currents are due to the regular magnetic field found by Babcock [12] at the polar regions of the sun. In fact Babcock's polar magnetic field has been found to be very variable in sign and intensity and even to reverse its directions [12,13]

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[10] H. Ellict: *Phil. Mag.*, 54, 601 (196).
[11] P. J. Coleman jr., L. Davis and C. P. Sonett: *Phys. Rev. Lett.*, **5**, 43 (1960).
[12] H. D. Babcock: *Ap. Journ.*, **130**, 364 (1959).
[13] M. Waldmeier: *Zeit. f. Ap.*, **49**, 176 (1960).

during a solar cycle; as a consequence DV's should be variable in amplitude and phase and have also a phase reversal, however during a period of observation of the cosmic ray intensity greater than a solar cycle such a variability of the DV has not been found ([14]). But if we can explain a train of DV's with a mechanism located near the sun or in the interplanetary space it remains to explain why the DV increases after the SSC (phenomenon located near the earth).

To solve this ambiguity we have to suppose there are at least two causes of the DV; we can suppose the second one geocentric and perhaps it can be a result of the configuration of the magnetosphere under the pressure of a solar plasma cloud.

* * *

We are very grateful to Mrs. M. Filippetti Ferrari for her help in the calculations.

We wish also to express our deep thanks to the persons in charge of the above mentioned laboratories whose cosmic ray data we have made use of. We wish also to thank the Japan National Committee for IGY for their « CR Intensity During IGY » part I, II, III, we have widely made use of for drawing those data.

([14]) V. Sarabhai, V. D. Desai and D. Venkatesan: *Phys. Rev.*, **99**, 1490 (1955).

RIASSUNTO

In questo lavoro si prendono in considerazione le depressioni che spesso precedono di meno di 24 ore i più importanti abbassamenti di Forbush, e certi aspetti di una tempesta di Forbush come la rarità degli inizi durante le ore del mattino, una sua maggiore ampiezza quando questi capitano durante la sera come pure certi lievoli aumenti che certe volte capitano durante la fase di diminuzione. Facciamo notare che una diminuzione di Forbush è sempre associata ad un treno di variazioni diurne e che viceversa un treno di variazioni diurne spesso contiene una diminuzione di Forbush più o meno evidente. Mediante un'analisi della dipendenza energetica dei suddetti preabbassamenti e la separazione delle variazioni locali dalle variazioni su scala mondiale che compaiono in un abbassamento di Forbush, si mostra che le suddette caratteristiche associate con un abbassamento, sono semplicemente dovute alla sovrapposizione della variazione diurna alla diminuzione di Forbush su scala mondiale. Il treno di variazioni diurne così separato dall'abbassamento su scala mondiale, ha una lunghezza variabile che apparentemente non supera la decina di giorni; è rinforzato e anticipato di fase durante un periodo di due o tre giorni comprendente la tempesta di Forbush.

IL NUOVO CIMENTO Vol. XXI, N. 6 16 Settembre 1961

On the Exact Theory of the Slowing-Down of Neutrons in an Infinite Homogeneous Medium.

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(ricevuto il 18 Maggio 1961)

Summary. — The object of this paper is to study the stationary energy distribution of neutrons slowed down in an infinite homogeneous absorbing medium. In particular, by confining the discussion to the field of the real variable, the following two problems will be considered: 1) The existence, for any scattering and capture law, of a formally exact solution of the Boltzmann equation; 2) The asymptotic behavior of this solution. By assuming a constant scattering cross-section, some details are given for the three practical cases of slowing down: without capture, with constant capture and with $1/v$-capture. For the last case, curves representing the exact solution evaluated numerically are shown.

1. – Introduction.

In the present paper, we shall discuss a new method for integrating the space-time independent Boltzmann equation governing the slowing-down of neutrons in an infinite homogeneous medium, in which the scattering and capture processes take place with any law of dependence on energy.

From a certain point of view, we shall give a more complete and general application of the new variable, the « relative lethargy », which we introduced in a previous paper [1].

(*) On leave of absence from Comitato Nazionale per l'Energia Nucleare, Laboratorio di Fisica Nucleare Applicata, Rome, Italy.

(1) V. C. Boffi: *Ann. Phys.*, **9**, 435 (1960).

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In particular, by confining ourselves to the field of this real variable and without making use of any transform, we shall consider the following two problems:

1) The existence, for any scattering and capture law, of an exact solution of the Boltzmann equation.

2) The asymptotic behavior of this solution.

Both these problems are formally solved in general in Sections 2 and 3 respectively. Some details will be given, however, for the most practical cases with constant capture and with $1/v$-capture, the scattering cross-section being constant.

For the case of $1/v$-capture, we numerically construct the exact solution and briefly discuss the corresponding curves in Appendix I (see Fig. 1).

2. – The formally exact solution.

2·1. *The integral equations of the slowing down process in terms of the relative lethargy.* – Let us consider an infinite homogeneous medium in which neutrons are uniformly produced with energy E_0 at a constant rate. Let $\Phi(E)$ be the non-virgin flux per unit energy at energy E and

$$F(E) = [\Sigma_s(E) + \Sigma_a(E)]\, \Phi(E)\,, \tag{1}$$

the total collision density per unit energy, where $\Sigma_s(E)$ and $\Sigma_a(E)$ are the macroscopic scattering and absorption cross-sections, respectively.

If a neutron collides only with point nuclei of mass number A which are free and at rest, the neutron balance equation can be written, in the space-time independent case, as [2]

$$\begin{cases} F(E) = \dfrac{h(E_0)}{E_0(1-\alpha)} + \dfrac{1}{1-\alpha}\displaystyle\int_E^{E_0} h(E')\,F(E')\,\frac{\mathrm{d}E'}{E'} & \text{for } \alpha E_0 < E < E_0\,; \\ F(E) = \dfrac{1}{1-\alpha}\displaystyle\int_E^{E/\alpha} h(E')\,F(E')\,\frac{\mathrm{d}E'}{E'} & \text{for } E < \alpha E_0\,, \end{cases} \tag{2}$$

[2] S. Glasstone and M. C. Edlund: *The Elements of Nuclear Reactor Theory* (New York, 1952).

where

$$h(E) = \frac{\Sigma_s(E)}{\Sigma_s(E) + \Sigma_a(E)}, \tag{3}$$

is the probability of scattering and

$$\alpha = \left(\frac{A-1}{A+1}\right)^2. \tag{4}$$

In the following, we shall assume that

$$0 < \alpha < 1. \tag{5}$$

The two cases, $\alpha = 0$ (hydrogen) and $\alpha = 1$ (ideal element of infinite mass), will be regarded as special cases and are discussed in Appendix II by considering both sides of eqs. (2) as continuous functions of α.

Now multiply both sides of eqs. (2) by E and introduce the relative lethargy [1]

$$x = \frac{\ln E_0/E}{\ln 1/\alpha}. \tag{6}$$

This new variable takes the integer value n when E is $\alpha^n E_0$, which is the energy of source neutrons which lose the maximum possible energy in successive collisions.

If we also put

$$EF(E) = E_0 \alpha^x F(x) = N(x), \tag{7}$$

eqs. (2) become

$$\begin{cases} N(x) = \dfrac{h(0)}{1-\alpha}\,\alpha^x + q\displaystyle\int_0^x \alpha^{x-y} h(y) N(y)\,\mathrm{d}y & \text{for} \quad 0 < x < 1; \\ N(x) = q\displaystyle\int_{x-1}^x \alpha^{x-y} h(y) N(y)\,\mathrm{d}y & \text{for} \quad x > 1, \end{cases} \tag{8}$$

where now $N(x)$ is just the total collision density per unit relative lethargy and

$$q = \frac{1}{1-\alpha} \ln \frac{1}{\alpha} \tag{9}$$

is a quantity characteristic of the medium and satisfies the inequality

$$1 < q < 1/\alpha \,, \tag{10}$$

when eq. (5) holds.

Eqs. (8) are Volterra linear integral equations of the second kind, the second one being homogeneous and having both the limits variable.

2·2. *First form of the exact solution.* – We shall now show that it is always possible to construct a formally exact solution of eqs. (8) by means of a step-by-step procedure, when $h(x)$ is any regular function of x.

First of all, we have to deal with the analytical nature of the unknown function, $N(x)$.

Actually, as the physical process requires, the points of integer abscissa $x = 1, 2, 3, \ldots$ are points of discontinuity either for the function $N(x)$ or for some of its derivatives. They belong to a denumerable set of measure zero (in the Lebesgue sense) so that the unknown function $N(x)$ is an almost continuous function.

Therefore, if we call $N_n(x)$ the solution of the proposed equations in the interval $(n, n+1)$, we can write the second of eqs. (8) as [1]

$$N_n(x) = q \int_{x-1}^{n} \alpha^{x-y} h(y) N_{n-1}(y)\, \mathrm{d}y + q \int_{n}^{x} \alpha^{x-y} h(y) N_n(y)\, \mathrm{d}y \,, \qquad (n = 1, 2, 3, \ldots) \,, \tag{11}$$

where the integral

$$S_n(x) = q \int_{x-1}^{n} \alpha^{x-y} h(y) N_{n-1}(y)\, \mathrm{d}y \,, \tag{12}$$

represents the source term for the interval $(n, n+1)$ and takes into account the rate at which neutrons are scattered from the interval $(n-1, n)$.

Thus, the second of eqs. (8), which governs the slowing-down process for $x > 1$, is actually an inhomogeneous Volterra linear integral equation of the second kind. Furthermore, the initial conditions are explicitly contained in this equation.

In fact, we get from the first of eqs. (8) and from eq. (11)

$$\begin{cases} N_1(1) = N_0(1) - \dfrac{\alpha}{1-\alpha} h(0) & \text{for} \quad n = 1; \\ N_n(n) = N_{n-1}(n) & \text{for} \quad n > 1 \,. \end{cases} \tag{13}$$

Eqs. (13) say that the unknown function $N(x)$ has a single discontinuity of the first kind at the point $x=1$ and some of its derivatives will have discontinuities of the first kind at the point $x=1, 2, 3, \ldots$ Thus, $N(x)$ is a piecewise continuous function.

By taking into account eq. (12), we may rewrite eq. (11) as

$$N_n(x) = S_n(x) + q\int_n^x K(x, y) N_n(y)\,\mathrm{d}y\,, \tag{14}$$

where

$$K(x, y) = \alpha^x[\alpha^{-y} h(y)] \tag{15}$$

is the associated kernel.

Eq. (14), which is valid for $n = 0, 1, 2, \ldots$ with

$$S_0(x) = \frac{h(0)}{1-\alpha}\,\alpha^x\,, \tag{16}$$

can be solved in two different ways.

First, one observes that the kernel is, in general, of the form

$$K(x, y) = k(x)/k(y)\,, \tag{17}$$

so that eq. (14) can be reduced to a first-order linear differential equation with the initial conditions given by eqs. (13).

We prefer, however, to solve eq. (14) by means of the classic method of the iterated kernels.

If we introduce the resolvent kernel

$$-H(x, y; q) = \sum_{m=0}^{\infty} q^m K_{m+1}(x, y)\,, \tag{18}$$

where

$$K_{m+1}(x, y) = \int_y^x K(x, z) K_m(z, y)\,\mathrm{d}z \qquad (m = 0, 1, 2 \ldots)\,, \tag{19}$$

with

$$K_1(x, y) \equiv K(x, y)\,, \tag{20}$$

then, as is well known from the theory of integral equations, the solution of

eq. (14) is

$$N_n(x) = S_n(x) - q\int_n^x H(x, y; q) S_n(y)\,dy \tag{21}$$

for $n = 0, 1, 2 \ldots$

Eq. (21) thus yields, step-by-step, the formally exact solution of the slowing-down process for any regular function $h(x)$ and for the actual initial conditions.

The iterated kernels, given by eq. (19), are physically related with the spectrum of the neutrons which have suffered $m = 0, 1, 2, \ldots$ collisions.

Whether it is possible to evaluate the solution of eq. (21) by simple quadratures will depend on the analytical form of $h(x)$.

For a constant macroscopic scattering cross-section, two cases are of practical interest:

1) Slowing down with constant capture,

$$h(x) = \text{const} = h; \qquad -H(x, y; qh) = h \exp\left[\left(qh - \ln\frac{1}{\alpha}\right)(x - y)\right]. \tag{22a}$$

(When $h = 1$, we have the particular case of slowing-down without capture.)

2) Slowing down with $1/v$-capture,

$$\begin{cases} h(x) = \dfrac{h_0 \alpha^{x/2}}{1 + h_0 \alpha^{x/2}} & \text{with} \qquad h_0 = \Sigma_s/\Sigma_a\,(0); \\ -H(x, y; q) = K(x, y)\left(\dfrac{1 + h_0 a^{y/2}}{1 + h_0 \alpha^{x/2}}\right)^{\mu} & \text{with} \qquad \mu = \dfrac{2}{1 - \alpha}. \end{cases} \tag{22b}$$

The first case does admit a solution, eq. (21), which can be evaluated by simple quadratures. For the case of $1/v$-capture, the exact solution cannot be expressed analytically, but can be evaluated only numerically from eqs. (11) or (21) (see Fig. 1).

2·3. *Second form of the exact solution.* – Eqs. (11) or (21) are very useful for the numerical evaluation of the exact solution. But, in order to study its asymptotic behavior, we need a different expression for the functions $N_n(x)$.

First, for the interval (0, 1), we have by differentiating the first of eqs. (8)

$$N_0'(x) = \left[qh(x) - \ln\frac{1}{\alpha}\right] N_0(x)\,, \tag{23}$$

from which, in general, the solution in the interval (0, 1) follows

$$N_0(x) = \frac{h(0)}{1-\alpha}\,\alpha^x \exp\left[q\int_0^x h(y)\,dy\right] = -\text{const}\,H(x, 0; q)\,. \tag{24}$$

In the case with constant capture, eq. (24) becomes

$$N_0(x) = \frac{h}{1-\alpha}\exp\left[\left(qh - \ln\frac{1}{\alpha}\right)x\right] \tag{25a}$$

and its behavior depends on the value of $h(h \gtrless 1-\alpha)$. When $h=1$, $N_0(x)$ is always increasing with x and has neither maxima nor minima except for $x=\infty$.

In the case with $1/v$-capture, eq. (24) reads as

$$N_0(x) = \frac{h(0)}{1-\alpha}\,(1+h_0)^\mu \frac{\alpha^x}{(1+h_0\alpha^{x/2})^\mu}\,. \tag{25b}$$

Now this solution shows a maximum at the point of abscissa

$$x_{\max} = \frac{2}{\ln(1/\alpha)}\ln\frac{\alpha}{r(1-\alpha)}\,, \tag{26}$$

where we have written $h_0 = 1/r$. The larger is r, the more amplified will be the effect of absorption (see Fig. 1). The value of $x_{\max}$, given by eq. (26), for any α decreases by increasing r; for any r, it increases by increasing α. When $r=\alpha/(1-\alpha)$, the maximum is at $x=0$ and for $r>\alpha/(1-\alpha)$ one has $x_{\max}<0$. For deuterium, one obtains practically $0<x_{\max}<1$.

Differentiating both the sides of eq. (11) with respect to x gives

$$N_n'(x) = \left[qh(x) - \ln\frac{1}{\alpha}\right]N_n(x) - \alpha q h(x-1)N_{n-1}(x-1)\,. \tag{27}$$

The differentiation is permitted by the continuity and regularity of the functions $N_n(x)$.

Dividing both the sides of eq. (27) by $N_0(x)$ and using eq. (23), one obtains the dimensionless functional equation

$$P_n'(x) = -A(x)\,P_{n-1}(x-1)\,, \tag{28}$$

where

$$P_n(x) = \frac{N_n(x)}{N_0(x)} \qquad \text{with} \qquad P_0(x) = 1 \qquad \text{for} \qquad n = 1, 2, 3, \ldots; \tag{29}$$

$$A(x) = \alpha q h(x-1) \frac{N_0(x-1)}{N_0(x)} = q h(x-1) \exp\left[-q \int_{x-1}^{x} h(y)\,\mathrm{d}y\right]. \tag{30}$$

The explicit form of $A(x)$ is

$$A(x) \equiv \text{const} = \lambda = qh \exp[-qh] \qquad \left(0 < \lambda < \frac{1}{e}\right) \tag{31a}$$

in the case with constant capture, and

$$A(x) = q \exp[-q] h(x-1) \left[\frac{h(x-1)}{h(x)}\right]^{\mu} = \frac{1}{[h_c(x)]^{\mu}} \frac{\mathrm{d}}{\mathrm{d}x} [h_c(x-1)]^{\mu} = \tag{31b}$$

$$= q \exp[-q] \frac{h_0 \alpha^{x/2}}{\alpha^{1/2} + h_0 \alpha^{x/2}} \left(\frac{1 + h_0 \alpha^{x/2}}{\alpha^{1/2} + h_0 \alpha^{x/2}}\right)^{\mu} \qquad (0 < A(x) < q)$$

in the case with $1/v$-capture. Here $h_c(x) = 1 - h(x)$ is the absorption probability.

The solution of eq. (28) is explicitly

$$P_n(x) = C_n - \int_{n}^{x} A(y)\, P_{n-1}(y-1)\,\mathrm{d}y\,, \qquad (n = 0, 1, 2, \ldots)\,, \tag{32}$$

where the integration constant C_n can be determined from the initial conditions. From eqs. (13) we have

$$C_1 = 1 - \frac{\alpha}{1-\alpha} \frac{h(0)}{N_0(1)} \qquad \text{for} \qquad n = 1; \tag{33}$$

$$C_n \equiv P_n(n) = P_{n-1}(n) \qquad \text{for} \qquad n > 1\,.$$

By recurrence, we get from eq. (32)

$$P_n(x) = \sum_{k=0}^{n} (-1)^k C_{n-k} J_{k,n}(x)\,, \tag{34}$$

where the function

$$(35)\quad J_k(x;\, n,\, n-1,\, \ldots,\, n-k+1) \equiv J_{k,\,n}(x) = \int_n^x A(y_n)\,\mathrm{d}y_n \int_{n-1}^{y_n-1} A(y_{n-1})\,\mathrm{d}y_{n-1} \ldots \int_{n-k+1}^{y_{n-k+2}-1} A(y_{n-k+1})\,\mathrm{d}y_{n-k+1}$$

represents the k-fold integral of the function $A(x)$ with $k = 1, 2, \ldots, n$ $(n \geqslant k)$ and the coefficients C_{n-k} are given by the following recursion formula:

$$(36)\quad C_{n-k} = \sum_{m=0}^{n-k-1} (-1)^m\, C_{n-k-m-1} J_{m,\,n-k-1}(n-k)\,.$$

Some properties of $J_{k,\,n}(x)$ are:

$$(37)\quad \begin{cases} J_{0,n}(x) = 1\,; \qquad J_{-k,n}(x) = 0\,; \\ J_{k,n}(n) = 0\,; \qquad J_{n,n}(n) = 0\,; \\ J^{(m)}_{k,n}(n) = 0 \quad \text{for } m = 0, 1, 2, \ldots, k-1\,; \\ J^{(k)}_{k,n}(n) = A(n)\,A(n-1) \ldots A(n-k+1) \quad \text{for } k = 1, 2, \ldots, n; \\ J'_{k,n}(x) = A(x) J_{k-1,\,n-1}(x-1)\,. \end{cases}$$

For the coefficients $C_n \equiv P_n(n)$, we have

$$(38)\quad C_n = \sum_{m=0}^{n-1} (-1)^m J_{m,m}(n) - h^* \sum_{m=0}^{n-2} (-1)^m J_{m,m+1}(n)\,,$$

where

$$(39)\quad h^* = \frac{\alpha}{1-\alpha}\,\frac{h(0)}{N_0(1)}\,.$$

We have now expressed the solution $N(x)$ in terms of the solution $N_0(x)$ by means of

$$(40)\quad N_n(x) = N_0(x)\,P_n(x) \qquad (n = 0, ,1\ , 2, \ldots)\,.$$

Introducing the variable

$$(41)\quad z = x - n\,, \qquad (n = 0, 1, 2, \ldots)$$

eq. (40) can be written as

$$N_n(z) = N_0(z+n)\,P_n(z) \tag{42}$$

with the identification

$$\begin{cases} N_n(z+n) = N_n(z)\ ; \\ P_n(z+n) = P_n(z)\ . \end{cases} \tag{43}$$

Thus, the functions $P_n(z)$ are all defined in the basic interval $(0, 1)$, and the solution $N(x) \equiv N_n(x)$ for $n \leqslant x \leqslant n+1$ is obtained by transferring the function $N_n(z)$, given by eq. (42), to the $(n+1)$-th interval along the x-axis.

Furthermore, by induction, one can derive the physically expected result

$$1 = P_0(z) > P_1(z) > P_2(z) > \dots\,, \tag{44a}$$

from which it also follows that

$$1 > C_1 > C_2 > C_3 > \dots\,. \tag{44b}$$

In principle, this implies the existence of a decreasing function $P(x)$ such that $\lim_{n\to\infty} P_n(x) = P(x)$ (see Section 3).

In the case of constant capture, the expressions of $P_n(x)$ are

$$P_n(x) = \sum_{k=0}^{n} (-1)^k C_{n-k} \lambda^k \frac{(x-n)^k}{k!}\,, \tag{45}$$

with

$$C_n = \sum_{m=0}^{n-1} (-1)^m (n-m)^m \frac{\lambda^m}{m!} = \exp[-q\lambda] \sum_{m=0}^{n-k-1} (-1)^m (n-m-1)^m \frac{\lambda^m}{m!}\,. \tag{46}$$

For $h = 1$, one obtains the corresponding expressions for the case without capture. No explicit expressions are available for the case of $1/v$-capture for the k-fold integrals $J_{k,n}(x)$, given by eq. (35), cannot be evaluated analytically.

3. – The asymptotic behavior of the exact solution.

3·1. *The case with constant capture.* – To study the asymptotic behaviour of the solution, one usually starts from the physical circumstance that the capture occurs at low energies, *i.e.*, for large x, and then neglects the contri-

bution of the interval $(0, 1)$. Thus, the second of eqs. (8) is used for all the values of x between 0 and ∞. No information is then left of the initial condition for the unknown function. Besides, if a continuous and regular solution $\bar{N}(x)$ of the second of eqs. (8) exists, one assumes that the actual solution takes everywhere the value of $\bar{N}(x)$. This implies:

1) The elimination of the discontinuity at $x=1$.

2) The physical condition that $\bar{N}(x) \geqslant 0$ for all x.

The larger the value of x, the better is the approximation of assuming $N(x) \equiv \bar{N}(x)$.

From a rigorous point of view, the problem of finding an asymptotic representation for the exact step-by-step solution consists of:

1) Constructing the function

$$CF(x) = \lim_{n\to\infty} N_n(x) \,. \tag{47}$$

By eq. (40) it is sufficient to study the limiting behavior of the $P_n(x)$'s only.

2) Showing that, if $F(x)$ exists and is positive, or at least not identically zero, it satisfies the second of eqs. (8). The constant C in eq. (47) is then the constant which completely defines the asymptotic behavior of the exact solution.

For the case with constant capture, this direct proof may be carried out in the following way.

We observe that the $P_n(x)$'s are polynomials in $(x-n)$ and rewrite eq. (45) as

$$P_n(x) = \sum_{k=0}^{n} (-1)^k C^*_{n-k} \lambda^k \frac{x^k}{k!} \,, \tag{48}$$

where

$$C^*_{n-k} = \sum_{m=0}^{n-k} (m+k)^m \frac{\lambda^m}{m!} - \exp[-qh] \sum_{m=1}^{n-k-1} (m+k+1)^m \frac{\lambda^m}{m!} \,. \tag{49}$$

Let us indicate by C^*_{asy} the value of C^*_{n-k} for large n.

From the theory of Lagrange inversion formula ([3]), we know that if η is a root of the trascendental equation

$$\eta = \lambda e^{\eta} \,, \tag{50}$$

([3]) E. T. Whittaker and G. N. Watson: *Modern Analysis* (Cambridge, 1958).

then, for any finite k,

$$(51) \qquad \sum_{m=0}^{\infty} (m+k)^m \frac{\lambda^m}{m!} = \frac{\exp[k\eta]}{1-\eta}\,; \qquad \sum_{m=0}^{\infty} (m+k+1)^m \frac{\lambda^m}{m!} = \frac{\exp[(k+1)\eta]}{1-\eta}\,,$$

provided $\lambda < 1/e$ as it is in the present case (see eq. (31a)). Then, for $n \to \infty$ eqs. (49) and (51) give

$$(52) \qquad C^*_{\text{asy}} = \frac{\exp[k\eta]}{1-\eta}\,(1 - \exp[-qh]\cdot\exp[\eta])\,.$$

Eq. (50) has the two roots

$$(53) \qquad \eta_1 = qh\,; \qquad \eta_2 = \alpha^\nu qh\,,$$

where ν is the real root of

$$(54) \qquad \frac{1-\alpha}{h} = \frac{1-\alpha^\nu}{\nu}\,,$$

which, for different purposes, we studied thoroughly in a previous paper [1].

The root $\eta_1 = qh$ gives $C^*_{\text{asy}} \equiv 0$. The other root gives

$$(55) \qquad C^*_{\text{asy}} = \frac{1-\alpha^\nu}{1-\alpha^\nu qh} \exp[\alpha^\nu qhk]$$

so that

$$(56) \qquad P_{\text{asy}}(x) = \frac{1-\alpha^\nu}{1-\alpha^\nu qh} \exp[-qh\alpha^\nu x]\,.$$

Substituting eq. (56) into the right-hand side of eq. (40) and using eq. (25a), we obtain

$$(57) \qquad N_{\text{asy}}(x) = \frac{\nu}{1-\alpha^\nu qh} \exp\left[-(1-\nu)\ln\frac{1}{\alpha}x\right].$$

When $h \to 1$, then $\nu = 1$ and eq. (57) becomes

$$(58) \qquad N_{\text{asy}}(x) = \text{const} = 1/\xi\,,$$

where

$$(59) \qquad \xi = 1 - \alpha q$$

is the mean logarithmic energy loss per collision.

One easily verifies that eqs. (57) and (58) satisfy the second of eqs. (8) when $h(x) = \text{const} = h$. Thus, we have proved that $\lim_{n\to\infty} N_n(x) = N_{\text{asy}}(x) \equiv \bar{N}(x)$, *i.e.*, the existence of the exact solution implies the existence of a positive solution of the second of eqs. (8), which represents the asymptotic behavior.

We now rewrite eq. (40) for $n \to \infty$ as

$$\bar{N}(x) = N_0(x) P(x) , \tag{60}$$

where $P(x)$ satisfies

$$P'(x) = - A(x) P(x-1) . \tag{61}$$

Then $P_{\text{asy}}(x)$, eq. (56), is solution of this equation with $A(x) = \lambda$ and also

$$\lim_{n\to\infty} P_n(x) = P_{\text{asy}}(x) \equiv P(x) . \tag{62}$$

A further consideration seems to be worthwhile at this point. One could also consider the sequence of the continuous functions, $N_n(x)$ (or $P_n(x)$), on the set of all the positive real numbers. This sequence may converge uniformly to a certain function $\bar{F}(x)$ not identically zero, which is the case having physical interest. From the definition of uniform convergence of a sequence of continuous functions [4], we know that corresponding to each arbitrarily chosen positive number ε, a number n_ε^* can be determined so that

$$| N_n(x) - N_m(x) | < \varepsilon \tag{63}$$

for every pair of integers $n, m > n_\varepsilon^*$ and for any x.

Then, if we specialize $n > m > n_\varepsilon^*$ and $m = n - 1$, eq. (63) implies that from a certain n^* on, we may write

$$N_n(x) \simeq N_{n-1}(x) \tag{64}$$

so that

$$\lim_{n\to\infty} N_n(x) = \bar{F}(x) . \tag{65}$$

Then eq. (11) reduces just to the homogeneous form given by the second of

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[4] E. W. HOBSON: *The Theory of Functions of a Real Variable*, Vol II (Cambridge, 1926).

eqs. (8) and $\bar{F}(x) = \bar{N}(x)$. But, even though one were able to prove eq. (63), it would not yet give information about the asymptotic behavior of the exact solution.

3·2. *The general case.* – Let us now rewrite the second of eqs. (8)

$$\bar{N}(x) = q \int_{x-1}^{x} \alpha^{x-y} h(y) \bar{N}(y)\, dy. \qquad (x > 1)\,. \tag{66}$$

Since we know the values of the unknown function $\bar{N}(x)$ in the whole interval (0, 1) from the first of eqs. (8), then, from the inhomogeneous form of eq. (11), we have seen that eq. (66) always admits a formally exact step-by-step solution.

When one wants to extend the validity of eq. (66) up to $x = 0$, this implies the study of the asymptotic behavior of the exact solution. This behavior is represented by the solution $\bar{N}(x)$ of eq. (66), as discussed in the preceding paragraph.

In order to do this, when, in eq. (66), $h(x)$ is any regular function of x, we start from the expectation that the procedure followed in the case of constant capture is still valid. That is, we should prove that the asymptotic representation of the $P_n(x)$'s, given by eq. (34), is solution of eq. (61) so that the solution $\bar{N}(x)$ of eq. (66) is given by eq. (60).

To evaluate $P_{\text{asy}}(x)$, let us put

$$\lim_{n\to\infty} J_{k,n}(x) = J_{k,\text{asy}}(x) \equiv J_k(x)\,. \tag{67}$$

If this limit exists for any k and is not identically zero, that is, if the sequence of the continuous functions $J_{k,n}(x)$ is, for any k, uniformly convergent, then, by eq. (64), the last of the properties of the $J_{k,n}(x)$'s in eq. (37) becomes, for $n \to \infty$,

$$J_k'(x) = A(x) J_{k-1}(x-1)\,. \tag{68}$$

Then

$$P_{\text{asy}}(x) \equiv P(x) = \sum_{k=0}^{\infty} (-1)^k J_k(x)\,, \qquad \text{with } J_0(x) = 1\,, \tag{69}$$

is easily verified to be a solution of eq. (61) for any regular $A(x)$.

From eq. (60) it follows that

$$\bar{N}(x) = N_0(x) \sum_{k-0}^{\infty} (-1)^k J_k(x)\,, \tag{70}$$

where $N_0(x)$ is given by eq. (24). Thus, $\bar{N}(x)$ is expressed in series of the asymptotic values of the k-fold integrals $J_{k,n}(x)$, eq. (35).

We must now show that eq. (70) is a solution of eq. (66). From eq. (61) with $P(x) = \bar{N}(x)/N_0(x)$, we obtain

$$\bar{N}'(x) + \ln\frac{1}{\alpha}\bar{N}(x) = qh(x)\bar{N}(x) - aqh(x-1)\bar{N}(x-1), \tag{71}$$

which is easily seen to be satisfied by eq. (70).

Multiplying both the sides of eq. (71) by α^{-x}, we get

$$\frac{\mathrm{d}}{\mathrm{d}x}[\alpha^{-x}\bar{N}(x)] = q\nabla[\alpha^{-x}h(x)\bar{N}(x)] = q\int_{x-1}^{x}\frac{\mathrm{d}}{\mathrm{d}y}[\alpha^{-y}h(y)\bar{N}(y)]\,\mathrm{d}y = \tag{72}$$

$$= q\,\frac{\mathrm{d}}{\mathrm{d}x}\int_{x-1}^{x}\alpha^{-y}h(y)\bar{N}\,(y)\,\mathrm{d}y,$$

where ∇ denotes the first receding difference, that is,

$$\nabla f(x) = f(x) - f(x-1)\,. \tag{73}$$

Integrating both the sides of eq. (72) with respect to x between 0 and x, we have

$$\bar{N}(x) = q\int_{x-1}^{x}\alpha^{x-y}h(y)\bar{N}(y)\,\mathrm{d}x + \alpha^{x}\left\{\bar{N}(0) - q\int_{-1}^{0}\alpha^{-y}h(y)\bar{N}(y)\,\mathrm{d}y\right\}, \tag{74}$$

which would coincide with eq. (66) only if

$$\bar{N}(0) = q\int_{-1}^{0}\alpha^{-y}h(y)\bar{N}(y)\,\mathrm{d}y\,. \tag{75}$$

By substituting eq. (70) into eq. (74), remembering the general expression of $N_0(x)$, eq. (24), and $A(x)$, eq. (30), and by iterated application of eq. (68), one may easily verify that both of eqs. (74) and (75) are satisfied so that eq. (70) is the general solution of eq. (66) for any regular function $h(x)$, for which eq. (67) is satisfied.

The explicit application of the method illustrated above depends, of course, on the knowledge of the $J_{k,n}(x)$'s, eq. (35), whose analytical evaluation by simple quadrature depends on the form of $A(x)$, *i.e.*, on $h(x)$.

We shall discuss this problem for the practical case of $1/v$-capture in the following paragraph.

Let us now make a last consideration. If we put in eq. (66)

$$\psi(x) = q\int_0^x \alpha^{-y} h(y)\bar{N}(y)\,\mathrm{d}y + \bar{C}\,, \tag{76}$$

we get

$$\bar{N}(x) = \alpha^x \nabla\psi(x)\,. \tag{77}$$

By differentiating with respect to x both the sides of eq. (76) and taking into account eq. (77), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}\psi(x) = \lambda(x)\nabla\psi(x)\,, \tag{78}$$

with $\lambda(x) = qh(x)$. The solution of such a mixed equation is a classic problem of the calculus of finite differences. We notice that $\psi(x) = \text{const}$ is always solution of eq. (78), but it yields for eq. (66) the trivial solution $\bar{N}(x) \equiv 0$.

Without entering into the discussion of eq. (78), we observe what follows.

If $\psi^*(x)$ is a solution of eq. (78), also

$$\psi(x) = (\bar{C} - C^*) + C^*\psi^*(x) \tag{79}$$

does. Now eq. (66) could be written as an inhomogeneous linear Volterra integral equation of the second kind, *i.e.*, as

$$\bar{N}(x) = S(x) + q\int_0^x K(x, y)\bar{N}(y)\,\mathrm{d}y\,, \tag{80}$$

where

$$S(x) = \alpha^x[\bar{C} - \psi(x-1)] \tag{81}$$

is a fictitious source which vanishes at $x = 1$, as expected. The kernel of eq. (80) is, of course, the same one appearing in the exact solution [see eq. (15)]. By means of eq. (70), we have $\psi(x)$ from eq. (76) and $S(x)$ from eq. (81). Then, eq. (80) represents the physical process for all the values of x between 0 and ∞ and corresponds to eq. (14) for the case of $n \to \infty$. For instance, in the case without capture, we get

$$S(x) = C^*(\alpha^x - \alpha) \tag{82}$$

with $C^* = 1/\xi(1-\alpha)$ and the solution of eq. (80) is just $\bar{N}(x) = 1/\xi$, as already seen in eq. (58).

3·3. *The case of* $1/v$*-capture.* – In the case of $1/v$-capture, the probability of scattering $h(x)$ is, remembering eq. (22*b*),

$$h(x) = \frac{\exp[-cx]}{r + \exp[-cx]}, \tag{83}$$

where

$$c = \frac{1}{2}\ln\frac{1}{\alpha}; \qquad r = 1/h_0 = \Sigma_a(0)/\Sigma_s. \tag{84}$$

Then $A(x)$ is given by eq. (31*b*) and it is not possible to evaluate the $J_{k,n}(x)$'s by simple quadrature (*).

One could solve the problem by the method proposed by SVARTHOLM (5), which consists in taking the Mellin transform of both sides of eq. (66). This method is just available only for the v^n-capture. It requires, in the present case, the change of variable $z = r\exp[cx]$ (the starting equation is then slightly different from Svartholm's) and leads also to a series solution.

We preferred, however, to construct the exact solution numerically from eq. (11) (see Appendix I) and to show the application of the method discussed in the preceding paragraph under the assumption that

$$h(x) \simeq \frac{1}{r}\exp[-cx]. \tag{85}$$

The larger x or r, the better is the approximation of replacing eq. (83) by eq. (85).

Then we have from eq. (24)

$$N_0(x) = \frac{\exp[q/rc]}{r(1-\alpha)}\exp[-2cx]\exp[(-q/rc)e^{-cx}] \simeq \frac{\exp[-2cx]}{(1-\alpha)} \tag{86}$$

(for large r),

and from eq. (30)

$$A(x) = \zeta\exp[-cx]\exp[-pe^{-cx}] = \tag{87}$$

$$= \lambda^*\frac{\mathrm{d}}{\mathrm{d}x}\exp[-pe^{-cx}] \simeq \zeta\exp[-cx] \quad \text{(for large } r\text{)},$$

(*) *Note added in proof.* – More recent investigations indicate that it may be possible to do this evaluation by simple quadrature.

(5) N. SVARTHOLM: *Acta Polytechnica*, No. 177 (1955).

where

$$(88)\qquad \begin{cases} p = \dfrac{q}{rc}(e^c - 1) = \dfrac{\mu}{r}\dfrac{1-\alpha^{1/2}}{\alpha^{1/2}}; \qquad \left(\mu = \dfrac{2}{1-\alpha}\right) \\ \zeta = \dfrac{q}{r}e^c = \dfrac{q}{r\alpha^{1/2}}; \\ \lambda^* = \zeta/pc = \dfrac{1}{1-\alpha^{1/2}} > 1\ . \end{cases}$$

We remark that p and ζ depend on r, *i.e.*, on the absorption, while λ^* is independent of r.

With $A(x)$ given by eq. (87), we are now able to evaluate the $J_{k,n}(x)$'s. We list only the first two values,

$$(89)\qquad \begin{cases} J_{1,n}(x) = \lambda^*(\exp[-pe^{-cx}] - \exp[-pe^{-cn}]); \\ J_{2,n}(x) = \lambda^{*2}\Big\{\dfrac{1}{1+e^c}\exp[-p(1+e^c)e^{-cx}] - \exp[-p(1+e^c)e^{-cn}] - \\ \qquad\qquad - \exp[-pe^{-c(n-1)}](\exp[-pe^{-cx}] - \exp[-pe^{-cn}])\Big\}, \end{cases}$$

which, for $n \to \infty$, are easily proved to satisfy eq. (68).

From eq. (89) we may write

$$(90)\qquad P_n(x) = \sum_{m=0}^{n} C^*_{n-k} F(k) \exp\Big[-pe^{-cx}\sum_{m=0}^{k-1} e^{mc}\Big]\ .$$

Furthermore, we get

$$(91)\qquad C^*_{\text{asy}} = \sum_{m=0}^{\infty} F_1(m)\frac{\lambda^{*m}}{m!} - h^*\sum_{m=0}^{\infty} F_2(m)\frac{\lambda^{*m}}{m!}\ ,$$

which, from a formula due to DARBOUX (6), can be written as

$$(92)\qquad C^*_{\text{asy}} = \exp[\lambda^*]G(\lambda^*)\ ,$$

where $G(\lambda^*)$ denotes an integral polynomial.

Apart from this multiplicative constant (which is just the constant which

(6) G. POLYA and G. SZEGÖ: *Aufgaben und Sehrsätze aus der Analysis* (Berlin, 1954).

completely defines the asymptotic behavior), we get from eq. (90) for $n \to \infty$

$$(93) \quad P_{asy}(x) \equiv P(x) = F(0) + F(1) \exp\left[-pe^{-cx}\right] + F(2) \exp\left[-p(1+e^{c})e^{-cx}\right] + \ldots =$$

$$= \sum_{k=0}^{\infty} F(k) \exp\left[-pe^{-cx} \sum_{m=0}^{k=1} e^{mc}\right],$$

where the coefficients

$$(94) \qquad F(k) = (-1)^k \frac{\lambda^{*k}}{\prod_{i=0}^{k-1} \left(\sum_{\xi=0}^{i} \exp\left[mc\right]\right)} \qquad \text{with} \qquad F(0) = 1,$$

depend only on the mass number and are independent of r.

Thus, the series on the right-hand side of eq. (93) is a series with alternating signs (see eq. (69)). The absolute ratio of two successive terms is

$$(95) \qquad \left| \frac{F(k+1)}{F(k)} \exp\left[-pe^{kc}\, e^{-cx}\right] \right|,$$

so that its convergence depends, for any fixed x, on the value of p, *i.e.*, on r. For $x = \infty$, we have

$$(96) \qquad P(\infty) = \sum_{k=0}^{\infty} F(k) = \sum_{k=0}^{\infty} (-1)^k \left\{ \frac{1}{\sum_{m=0}^{k(k-1)/2} f(m)\, e^{mc}} \right\}.$$

Eq. (61) is now

$$(97) \qquad P'(x) = -\zeta \exp\left[-cx\right] \cdot \exp\left[-pe^{-cx}\right] P(x-1)$$

and it is solved by the series, eq. (93), only if

$$(98) \qquad \frac{F(k+1)}{F(k)} = -\frac{\lambda^{*}}{\sum_{m=0}^{k} \exp\left[mc\right]} = -\frac{\alpha^{k/2}}{1-\alpha^{(k+1)/2}},$$

which is just satisfied by the expression of $F(k)$, given by eq. (94).

Finally, from eq. (60), apart from the multiplicative constant C^{*}_{asy}, eq. (92), it follows that

$$(99) \qquad \bar{N}(x) = \exp\left[-2cx\right] \cdot \exp\left[(-q/rc)e^{-cx}\right] \cdot \sum_{k=0}^{\infty} F(k) \exp\left[-pe^{-cx} \sum_{m=0}^{k-1} e^{mc}\right].$$

We shall now check that eq. (99) is solution of eq. (66).

By substituting eq. (99) into eq. (74) and assuming that, term by term, differentiation and integration of the series in eq. (99) are permissible, then, after some calculations, one gets

$$(100)\quad \bar{N}(x) = \exp[-2cx]\exp[(-q/rc)e^{-c}]\cdot\sum_{k=0}^{\infty}F(k)\exp\Big[-pe^{-cx}\sum_{m=0}^{k-1}e^{mc}\Big] -$$

$$-\exp[-2cx]\sum_{k-1}^{\infty}C(k,m)F(k-1)\left(\exp[(-q/rc)e^{kc}e^{-cx}] - \exp[(-q/rc)e^{kc}]\right),$$

where

$$(101)\quad C(k,m) = \sum_{m=0}^{\infty}(-1)^m\cdot\left(\frac{1-\alpha^{(k-1)/2}}{\alpha^{(k-1)/2}}\right)^m - \frac{1}{\alpha^{1/2}}\cdot\sum_{m=0}^{\infty}(-1)^m\cdot\left(\frac{1-\alpha^{k/2}}{\alpha^{k/2}}\right)^m.$$

But, from the well-known formula

$$(102)\qquad \sum_{m=0}^{\infty}(-1)^m\cdot z^m = \frac{1}{1+z},$$

one verifies easily that, for any k, the coefficients $C(k, m)$, eq. (101), are identically zero so that eq. (99) represents the solution of eq. (66) when $h(x)$ is given by eq. (85).

By taking into account that for any $n = 0, 1, 2, \ldots$, eq. (66) can be written as

$$(103)\qquad \frac{\mathrm{d}^n}{\mathrm{d}x^n}\bar{N}^{(n)}(x) \equiv \bar{N}^{(n)}(x) = q\int_{x-1}^{x}\alpha^{x-y}\cdot[h(y)\bar{N}(y)]^n\,\mathrm{d}y$$

(one can verify this result by integrating by parts the right-hand side of eq. (103)) and from the form of eq. (99), one concludes that, for the given $h(x)$ and $n = 0, 1, 2, \ldots$,

$$(104)\qquad [\bar{N}^{(n)}(x)]_{x=\infty} = 0,$$

i.e., the function $\bar{N}(x)$ has an essential singularity at infinity. It behaves asymptotically like $\exp[-2cx]$, *i.e.*, in terms of the usual lethargy, like $\exp[-u]$. This asymptotic behavior is thus independent of the magnitude of the absorption.

4. – Conclusions.

Starting from eq. (11), we demonstrated in general the existence of a formally exact step-by-step solution for the space-time independent Boltzmann equation governing the neutron slowing-down. The solution $N_n(x)$ in the general interval $(n, n+1)$ $(n = 0, 1, 2, ...)$ is given by eq. (21).

By expressing $N_n(x)$, as in eq. (40), we also proved that the asymptotic representation, eq. (70), of the exact solution satisfies the source-free Boltzmann equation, eq. (66), provided eq. (67) holds.

* * *

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The numerical calculations were also performed through the hospitality of General Atomic.

Appendix I.

For the case of $1/v$-capture, Fig. 1 shows the logarithm of the total collision density, $\bar{N}(x)$, *vs.* the relative lethargy, x, for deuterium, beryllium and graphite and for different values of r. The curves were constructed by numerical integration of eq. (27).

The parameter r, which measures the degree of poisoning in order to amplify the effects of the absorption, is given by

$$r = \Sigma_a(0)/\Sigma_s = \frac{N_a \sigma_a(0)}{N_s \sigma_s} = \frac{N_a}{N_s} \tag{A-I.1}$$

by normalizing to $\sigma_a(0)/\sigma_s = 1$. When $r = 0$, we have the Placzek's curves [7] for the case without capture.

In the case of deuterium, for instance, it is possible to notice the agreement,

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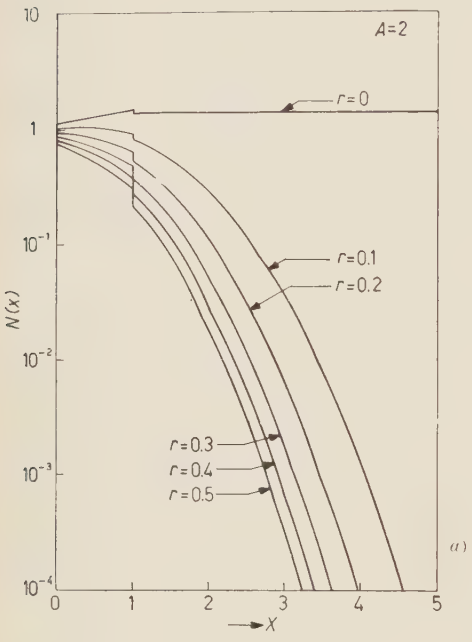
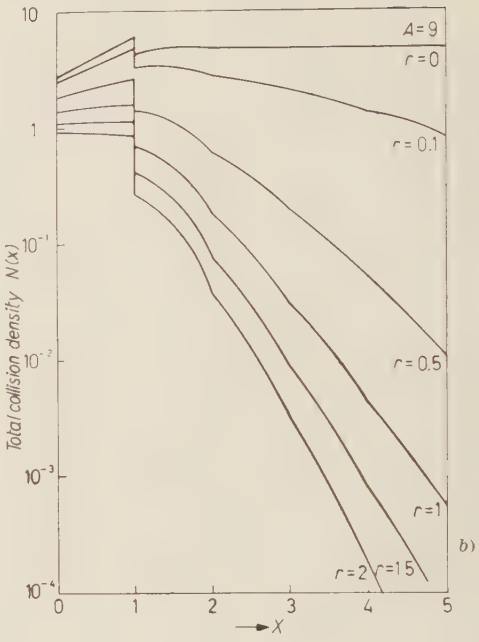
[7] G. Placzek: *Phys. Rev.*, **69**, 423 (1946).

for large r, between the numerical calculations and the solution, eq. (99), when $h(x)$ is given by the approximate expression, eq. (85).





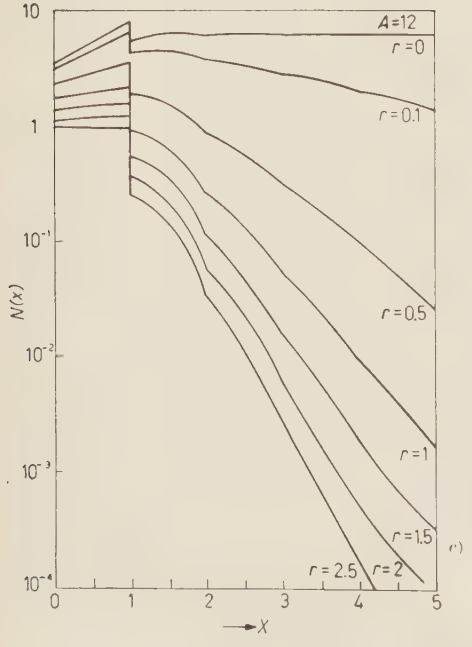


Fig. 1*a*), *b*), *c*). – The logarithm of the total collision density, $N(x)$, *vs.* the relative lethargy, x, for $A=2$, 9, 12.

APPENDIX II.

In the case of hydrogen, if we take the limit for $\alpha \to 0$ of both sides of eqs. (2), we get

$$\text{(A-II.1)} \qquad \begin{cases} F_H(E) = \dfrac{h(E_0)}{E_0} + \displaystyle\int_E^{E_0} h(E') F_H(E') \dfrac{\mathrm{d}E'}{E'} & \text{for} \quad 0 < E < E_0 ; \\ F_H(E) = \displaystyle\int_E^{\infty} h(E') F_H(E') \dfrac{\mathrm{d}E'}{E'} & \text{for} \quad E < 0 , \end{cases}$$

and it is easily seen that the second of these equations is obtained from the first one in the limit of $E_0 \to \infty$.

When E_0 is finite, we have for the solution of the first of eqs. (A-II.1) in the case of $1/v$-capture

$$\text{(A-II.2)} \qquad F_{H,E_0}(E) = \left(1 + \frac{h_0}{\sqrt{E_0}}\right) \frac{1}{(\sqrt{E} + h_0)^2} ,$$

while the solution of the second of eqs. (A-II.1) is

$$\text{(A-II.3)} \qquad \lim_{E_0 \to \infty} F_{H,E_0}(E) \equiv F_{H,\infty}(E) = \frac{1}{(\sqrt{E} + h_0)^2} .$$

When $\alpha = 1$, by applying the Hospital's rule, one gets from the second of eqs. (2), as physically expected,

$$\text{(A-II.4)} \qquad F(E) \equiv 0 .$$

RIASSUNTO (*)

Scopo di questo articolo è di studiare la distribuzione stazionaria dell'energia dei neutroni rallentati in un mezzo assorbente omogeneo infinito. In particolare, confinando la discussione al campo della variabile reale, considererò i due problemi seguenti: 1) l'esistenza, per ogni legge di scattering e di cattura, di una soluzione formalmente esatta dell'equazione di Boltzmann; 2) il comportamento asintotico di questa soluzione. Presupponendo la costanza della sezione d'urto per lo scattering, do alcuni dettagli in tre casi pratici di rallentamento: senza cattura, con cattura costante, con cattura secondo $1/v$. Per quest'ultimo caso, do le curve che rappresentano la soluzione esatta valutata numericamente.

(*) *Traduzione a cura della Redazione.*

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A Model for Non-leptonic Weak Interactions.

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Summary. — The suggestion is made that the non-leptonic weak interaction may be completely dissimilar to the leptonic weak interaction. On this assumption, a simple model for the non-leptonic weak interaction is given. The model is based on Pais' idea of a « parity clash », and is constructed to give the $|\Delta T| = \frac{1}{2}$ rule and the observed asymmetry parameters in pionic hyperon decays.

1. – Introduction.

Recently a number of weak interaction models have been proposed which contain the $|\Delta T| = \frac{1}{2}$ selection rule for non-leptonic decays ([1-3]). In all of these models, the absence of neutral leptonic currents in the interaction Hamiltonian has necessitated an abandonment of the (overall current) × (overall current) or (overall current) × (intermediate boson) pictures of the weak interaction in favor of a (leptonic interaction) + (non-leptonic interaction) structure. Since the $|\Delta T| = \frac{1}{2}$ rule has now become quite well established ([4]), it looks as

(*) Also supported by the Nuclear Research Foundation within the University of Sydney.

([1]) S. Treiman: *Nuovo Cimento*, **15**, 916 (1960).

([2]) A. Pais: *Nuovo Cimento*, **18**, 1003 (1960); *Phys. Rev.*, **122**, 317 (1961).

([3]) G. Takeda and M. Kato: *Prog. Theor. Phys. (Kyoto)*, **21**, 441 (1959); S. A. Bludman: *Phys. Rev.*, **115**, 468 (1959); T. D. Lee and C. N. Yang: *Phys. Rev.*, **119**, 1410 (1960); W. B. Zeleny and A. O. Barut: *Phys. Rev.*, **121**, 908 (1961); and others.

([4]) See, for example, R. Dalitz: *Rev. Mod. Phys.*, **31**, 823 (1959).

though this separation of the weak interaction Hamiltonian into leptonic and non-leptonic parts has become a necessity.

If we are to accept this separation, the question arises as to whether or not it is any longer desirable to pattern the non-leptonic part of the interaction after the leptonic part. Indeed, the current×current and V-A aspects of the currently popular weak interaction models were originally devised solely on the basis of leptonic interactions, and have not proven particularly convenient for describing non-leptonic decays. The suggestion which we wish to make here, then, is that perhaps this break in the weak interaction should be made complete, and the non-leptonic part examined in its own light. We propose a simple model for non-leptonic weak interactions which reproduces the known features of pionic hyperon decays. The model is patterned after the recent ideas of PAIS (2), and, indeed, is a fairly natural realization of Pais' « parity clash ».

2. – The interaction structure.

The outstanding features of non-leptonic decays which we wish to describe are:

I) the $|\Delta T| = \frac{1}{2}$ rule;

II) the asymmetry parameters $\alpha = 2\,\mathrm{Re}\,(S^*P)/(|S|^2+|P|^2)$, where S and P are the S-wave and P-wave amplitudes, respectively, in the pionic modes of hyperon decay.

The most striking aspects of II) are the large magnitudes of the asymmetry parameters for $\Sigma^+ \to \mathrm{p}+\pi^0$, $\Lambda \to \mathrm{p}+\pi^-$, and $\Lambda \to \mathrm{n}+\pi^0$ (we denote these asymmetry parameters by α^0_Σ, α^-_Λ, and α^0_Λ, respectively) and the small magnitudes of the asymmetry parameters for $\Sigma^+ \to \mathrm{n}+\pi^+$ and $\Sigma^- \to \mathrm{n}+\pi^-$ (denoted by α^+_Σ and α^-_Σ, respectively) (5). Probably the most successful attempt thus far to incorporate both I) and II) into a weak interaction structure has been the recent work of PAIS (2), and we shall borrow heavily from this work. In particular, we shall make use of the notion of a globally symmetric weak interaction and a parity clash.

The simplest of all possible non-leptonic weak interactions is the two-field interaction in which one particle is converted directly into another of different strangeness and isotopic spin. The most general such interaction (without

(5) See, for example, B. CORK, L. KERTH, W. A. WENZEL, J. W. CRONIN and R. L. COOL: *Phys. Rev.*, **120**, 1000 (1960).

derivatives) obeying $|\Delta S|=1$ is

$$H_{\text{weak}} = H_1 + H_2 ,$$

$$H_1 = [p(A_1 + iB_1\gamma^5)\Sigma^+ + \bar{n}(A_2 - iB_2\gamma^5)\Sigma^0 + \bar{n}(A_3 - iB_3\gamma^5)\Lambda + \bar{\Sigma}^-(A_4 + iB_4\gamma^5)\Xi^- + \bar{\Sigma}^0(A_5 + iB_5\gamma^5)\Xi^0 + \bar{\Lambda}(A_6 + iB_6\gamma^5)\Xi^0] + \text{Hermitian conjugate},$$

$$H_2 = (C_1\pi^+K^{+\dagger} + C_2\pi^0K^{0\dagger}) + \text{Hermitian conjugate},$$

where the A_i, B_i, and C_i are coupling constants. For CP invariance, we require the A_i, B_i, and C_i to be real (6). In order to obtain I), we must choose the coupling constants so that H_{weak} transforms as an isotopic spinor. In doing this, we shall write H_1 in a globally symmetric manner. Using the notation

$$N_1 = \begin{pmatrix} p \\ n \end{pmatrix}, \qquad N_2 = \begin{pmatrix} \Sigma^+ \\ Y^0 \end{pmatrix}, \qquad N_3 = \begin{pmatrix} Z^0 \\ \Sigma^- \end{pmatrix}, \qquad N = \begin{pmatrix} \Xi^0 \\ \Xi^- \end{pmatrix},$$

$$Y^0 = \sqrt{\tfrac{1}{2}}(\Lambda - \Sigma^0), \qquad Z^0 = \sqrt{\tfrac{1}{2}}(\Lambda + \Sigma^0),$$

we write

$$\text{(1)} \qquad \begin{cases} H_1 = [\bar{N}_1(f + ig\gamma^5)N_2 + \bar{N}_3(f + ig\gamma^5)N_4] + \text{H.c.}, \\ H_2 = h(\sqrt{2}\,\pi^+K^{+\dagger} - \pi^0K^{0\dagger}) + \text{H.c.}, \end{cases}$$

where, for simplicity, we have reduced the number of coupling constants to three. The results we shall obtain in this paper do not depend essentially on the local nature of our interaction Hamiltonian, and hence eq. (1) may be regarded, if we like, as a phenomenological description of a more complicated, non-local interaction.

For the strong interactions, we take

$$H_\pi = G_\pi \sum_{i=1}^{4} \bar{N}_i\gamma^5\boldsymbol{\tau}N_i\cdot\boldsymbol{\pi},$$

$$H_K = G_K(\bar{N}_1\gamma^5\tau_i\Sigma_iK + \bar{N}_1\gamma^5\Lambda K + \bar{N}_4\gamma^5\tau_i\Sigma_i\bar{K} + \bar{N}_4\gamma^5\Lambda\bar{K}) + \text{H.c.},$$

(6) We are using $\gamma^{5\dagger}=\gamma^{5*}=-\gamma^5$.

where

$$\pi_1 = \sqrt{\tfrac{1}{2}}(\pi^+ + \pi^-)\,, \qquad \pi_2 = \frac{i}{\sqrt{2}}(\pi^+ - \pi^-)\,, \qquad \pi_3 = \pi^0\,,$$

$$\Sigma_1 = \sqrt{\tfrac{1}{2}}(\Sigma^+ + \Sigma^-)\,, \qquad \Sigma_2 = \frac{i}{\sqrt{2}}(\Sigma^+ - \Sigma^-)\,, \qquad \Sigma_3 = \Sigma^0\,,$$

$$K = \begin{pmatrix} K^+ \\ K^0 \end{pmatrix}, \qquad \bar{K} = \begin{pmatrix} \bar{K}^0 \\ K^- \end{pmatrix}.$$

3. – Hyperon decays.

To the extent that global symmetry (7) holds for the strong π-baryon interactions, the lowest order diagrams for the known pionic modes of hyperon decay are shown in Fig. 1. It is immediately evident that in the global symmetry approximation, only H_2, which is necessarily parity conserving (8), contributes

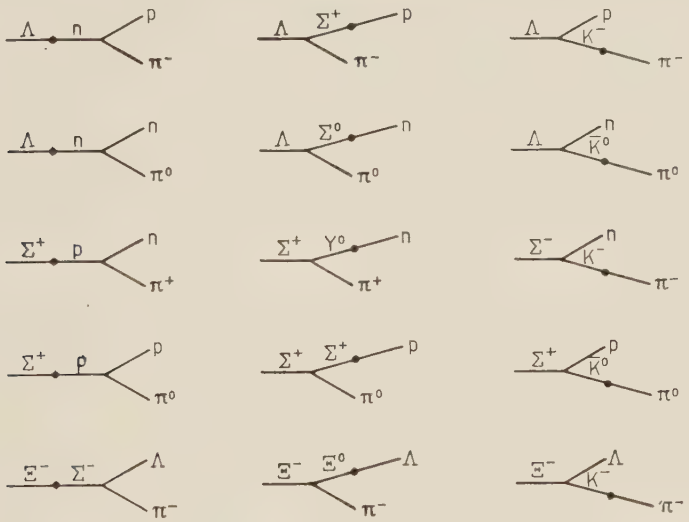

Fig. 1. – Lowest order diagrams for pionic hyperon decays.

to $\Sigma^- \to n + \pi^-$. Thus, in this approximation, $\Sigma^- \to n + \pi^-$ is pure P-wave, and $\alpha_\Sigma^- \approx 0$. Furthermore, to the extent that we also neglect final state interactions, only H_1 contributes to $\Sigma^+ \to n + \pi^+$. Thus, in order to obtain $\alpha_\Sigma^+ \approx 0$

(7) More precisely, we are assuming G^+ symmetry, in the notation of reference (3).
(8) We assume even K-π parity.

and a parity clash (2), we take

$$f/g \approx 0 \,. \tag{2}$$

The decay $\Sigma^+ \to n+\pi^+$ then proceeds primarily via S-wave.

An alternative description of the processes induced by H_1 and H_2 can be obtained through a diagonalization technique developed by CABIBBO and GATTO, FEINBERG, KABIR, and WEINBERG, and OKUBO (9). Taking

$$H_{\text{free}} = \bar{N}(i\partial_\mu\gamma^\mu + m_N)N + (\partial_\mu B^\dagger)(\partial^\mu B) + B^\dagger \varrho B \,,$$

where

$$N = \begin{pmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{pmatrix}, \qquad B = \begin{pmatrix} \pi^- \\ K^- \\ \sqrt{\tfrac{1}{2}}\,\pi^0 \\ \sqrt{\tfrac{1}{2}}\,K_2^0 \end{pmatrix}, \qquad \varrho = \begin{pmatrix} m_\pi^2 & & & \\ & m_K^2 & & \\ & & m_\pi^2 & \\ & & & m_K^2 \end{pmatrix},$$

we can diagonalize $H_{\text{free}} + H_1 + H_2$ so that it does not connect different fields. Assuming f to be at most of order g^2/m_N, we then obtain new fields (denoted by primes) given, to first order in the weak coupling, by

$$N = S_N N' \,, \qquad B = S_B B' \,, \tag{3}$$

where

$$S_N = \left(\begin{array}{cc|cc} 1 & -\lambda i\gamma^5 & & \\ \lambda i\gamma^5 & 1 & \multicolumn{2}{c}{0} \\ \hline & & 1 & -\lambda i\gamma^5 \\ \multicolumn{2}{c|}{0} & \lambda i\gamma^5 & 1 \end{array}\right),$$

$$S_B = \left(\begin{array}{cc|cc} 1 & \beta & & \\ -\beta & 1 & \multicolumn{2}{c}{0} \\ \hline & & 1 & -\beta \\ \multicolumn{2}{c|}{0} & \beta & 1 \end{array}\right),$$

$$\lambda = g/2m_N \,, \qquad \beta = \sqrt{2}\,h/(m_K^2 - m_\pi^2) \,.$$

Placing eq. (3) in H_π and H_K, we obtain, again to first order in the weak

(9) N. CABIBBO and R. GATTO: *Phys. Rev.*, **116**, 1334 (1959); G. FEINBERG, P. KABIR and S. WEINBERG: *Phys. Rev. Lett.*, **3**, 527 (1959); S. OKUBO: *Nuovo Cimento*, **16**, 963 (1960).

coupling,

$$(4)\qquad H_\pi = G_\pi \bar{N}'\gamma^5\boldsymbol{\tau}N'\cdot\boldsymbol{\pi}' + 2i\lambda G_\pi(\bar{N}_1'\boldsymbol{\tau}N_2'\cdot\boldsymbol{\pi}' + \bar{N}_3'\boldsymbol{\tau}N_4'\cdot\boldsymbol{\pi}' - \text{H. c.}) + \tilde{H}_\pi\,,$$

$$(5)\qquad \begin{cases} H_K = G_K(\bar{N}_1'\gamma^5\tau_i\Sigma_i'K' + \bar{N}_1'\gamma^5\Lambda'\bar{K}' + \bar{N}_4'\gamma^5\tau_i\Sigma_i'K' + \bar{N}_4'\gamma^5\Lambda'\bar{K}' + \text{H. c.}) - \\ -\beta G_K(\sqrt{2}\,\bar{n}'\gamma^5\Sigma^{-\prime}\pi^{+\prime} - \bar{p}'\gamma^5\Sigma^{+\prime}\pi^{0\prime} + \bar{p}'\gamma^5\Sigma^{0\prime}\pi^{+\prime} + \\ +\sqrt{\tfrac{1}{2}}\,\bar{n}'\gamma^5\Sigma^{0\prime}\pi^{0\prime} + \bar{p}'\gamma^5\Lambda'\pi^{+\prime} - \sqrt{\tfrac{1}{2}}\,\bar{n}'\gamma^5\Lambda'\pi^{0\prime} + \\ +\sqrt{2}\,\bar{\Xi}^{0\prime}\gamma^5\Sigma^{+\prime}\pi^{-\prime} + \bar{\Xi}^{-\prime}\gamma^5\Sigma^{-\prime}\pi^{0\prime} + \sqrt{\tfrac{1}{2}}\,\bar{\Xi}^{0\prime}\gamma^5\Sigma^{0\prime}\pi^{0\prime} - \\ -\bar{\Xi}^{-\prime}\gamma^5\Sigma^{0\prime}\pi^{-\prime} + \sqrt{\tfrac{1}{2}}\,\bar{\Xi}^{0\prime}\gamma^5\Lambda'\pi^{0\prime} + \bar{\Xi}^{-\prime}\gamma^5\Lambda'\pi^{-\prime} + \text{H. c.}) + \tilde{H}_K\,, \end{cases}$$

where $K^{0\prime} \equiv \sqrt{\tfrac{1}{2}}\,(K_1^0 + K_2^{0\prime})$ and $\bar{K}^{0\prime} = \sqrt{\tfrac{1}{2}}\,(K_1^0 - K_2^{0\prime})$. The interactions $\tilde{H}_\pi$ and $\tilde{H}_K$ are of orders $G_\pi\lambda$ and $G_K\lambda$, respectively, and do not involve the π' fields. Hence they do not contribute to pionic hyperon decays except at higher orders in G_π and G_K.

The interaction $H_\pi + H_K + H_1 + H_2$ has now been effectively transformed into the interactions (4) and (5). These give rise directly to pionic hyperon decays in the first order in the weak and strong couplings. Like the original interactions, they give $|\Delta T| = \frac{1}{2}$ for these decays. Eq. (4) gives rise to S-wave decays and eq. (5) to P-wave decays. Furthermore, to order $G_\pi\lambda$, eq. (4) contributes to all pionic hyperon decays except $\Sigma^- \to \mathrm{n} + \pi^-$, and to order $G_K\beta$, eq. (5) contributes to all except $\Sigma^+ \to \mathrm{n} + \pi^+$. Thus, to these orders, the gross features of the observed parity properties of pionic hyperon decays follow immediately.

The magnitudes of g and h can now be fitted to the experimental decay rates for $\Sigma^\pm \to \mathrm{n} + \pi^\pm$. Using the diagrams of Fig. 1, simple perturbation calculations indicate

$$|G_\pi g| \approx 1.6\cdot 10^7 \text{ cm}^{-1}$$

$$|G_K h| \approx 1.1\cdot 10^{21} \text{ cm}^{-2}$$

in natural units ($c = \hbar = 1$). Perturbation calculations also yield the decay rates, Γ, and asymmetry parameters listed in Table I. These figures are, of course, only rough estimates, because of the use of perturbation theory. The rather close fit in decay rates is to be expected, since the differences in the experimental pionic hyperon decay rates are almost completely accounted for by the phase space factors and the $|\Delta T| = \frac{1}{2}$ rule.

Experimentally, the only asymmetry parameter whose sign has been observed is α_Λ^-, and the situation here is still unsettled, recent work by Birge

TABLE I. – *Decay rates and asymmetry parameters for pionic hyperon decays.*

	$\Gamma_{\text{theo}} \cdot 10^{-9}$ (s^{-1})	$\Gamma_{\text{exp}} \cdot 10^{-9}$ (s^{-1})	$\alpha_{\text{theo}}(a)$	$\lvert\alpha\rvert_{\text{exp}}$
$\Lambda \to p+\pi^-$	2.4	$2.42 \pm .25$ (*b*)	$-.77\varepsilon$	$\geqslant .73 \pm .14$ (*c*)
$\Lambda \to n+\pi^0$	1.2	$1.42^{+.21}_{-.19}$ (*b*)	$-.77\varepsilon$	$\geqslant .60 \pm .13$ (*f*)
$\Sigma^+ \to n+\pi^+$	—	6.7 ± 1.1 (*e*)	0	$< .04 \pm .11$ (*f*)
$\Sigma^+ \to p+\pi^0$	6.0	6.7 ± 1.1 (*e*)	$+.99\varepsilon$	$\geqslant .75 \pm .17$ (*f*)
$\Sigma^- \to n+\pi^-$	—	$5.81^{+.36}_{-.52}$ (*d*)	0	$.14 \pm .20$ (*g*)
$\Xi^- \to \Lambda+\pi^-$	2.8	7.81 ± 1.90 (*h*)	$-.92\varepsilon$	$.69 \pm .36$ (*i*)

(*a*) ε is the sign of $G_\pi g/G_K h$.
(*b*) *1958 Annual International conference on High Energy Physics at CERN*, edited by B. FERRETTI (Geneva, 1958), pp. 270, 273.
(*c*) *1958 Annual International Conference on High Energy Physics at CERN*, edited by B. FERRETTI (Geneva, 1958), pp. 267, 323.
(*d*) *1958 Annual International Conference on High Energy Physics at CERN*, edited by B. FERRETTI (Geneva, 1958), p. 271.
(*e*) *1958 Annual International Conference on High Energy Physics at CERN*, edited by B. FERRETTI (Geneva, 1958), p. 271. The branching ratio $\Gamma(\Sigma^+ \to p+\pi^0)/\Gamma(\Sigma^+ \to n+\pi^+) = 1.00 \pm 0.09$ is taken from S. C. FREDEN, H. N. KORMBLUM and R. S. WHITE: *Nuovo Cimento*, **16**, 611 (1960).
(*f*) See reference ([5]).
(*g*) See reference ([5]). This value is obtained on the assumption that $|\Delta T| = \frac{1}{2}$ holds for pionic Σ-decays.
(*h*) See reference ([12]).
(*i*) See reference ([12]). This value is based on the value $|\alpha_\Lambda^-| = .95 \pm .07$.

and FOWLER ([10]) being in conflict with earlier work by BOLDT *et al.* ([11]). In our model, the sign of α_Λ^- is $-\varepsilon$, where ε is the sign of $G_\pi g\, G_K h$. Like Pais' model ([2]), our model predicts that α_Σ^0 and α_Λ have opposite sign. The relative sign of the asymmetry parameters for Λ and Ξ-decays is actually not determined by this model. It can be made negative by changing the relative sign of the two terms in H_1. This would bring it into agreement with the recent experimental evidence of FOWLER *et al.* ([12]).

4. – K-meson decays.

Little information can be obtained from this model on non-leptonic K-meson decays, except that they obey $|\Delta T| = \frac{1}{2}$ and can proceed via such diagrams

([10]) R. W. BIRGE and W. B. FOWLER: *Phys. Rev. Lett.*, **5**, 254 (1960).
([11]) E. BOLDT, H. S. BRIDGE, D. O. CALDWELL and Y. PAL: *Phys. Rev. Lett.*, **1**, 256 (1958).
([12]) W. B. FOWLER, R. W. BIRGE, P. EBERHARD, R. ELY, M. L. GOOD, W. M. POWELL and H. K. TICHO: *Phys. Rev. Lett.*, **6**, 134 (1961).

as those of Fig. 2. The decay $K^+ \to \pi^+ + \pi^0$ is, of course, forbidden by the $|\Delta T| = \frac{1}{2}$ rule. Therefore, there may be some additional baryon terms in H_{weak} which are not isotopic spinors. If we assume, as suggested by TREIMAN (1) and PAIS (2), that there exists a connection between global symmetry and the $|\Delta T| = \frac{1}{2}$ rule, then presumably these additional terms are also not globally symmetric, and are coupled with a coupling constant of the order of $\frac{1}{10}\, g$, giving rise to the well-known violation of the $|\Delta T| = \frac{1}{2}$ rule.

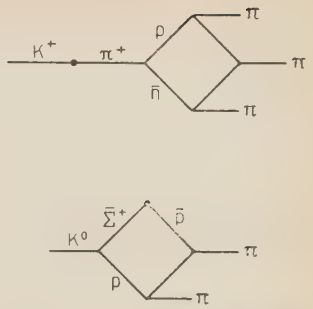

Fig. 2. – Possible diagrams for non-leptonic K-decays.

5. – Conclusion.

Barring dynamical accidents, the most promising means of describing the $|\Delta T| = \frac{1}{2}$ rule and the known asymmetry parameters of pionic hyperon decays seems to be through a globally symmetric weak interaction (1,2) and a parity clash (2). The special advantage of the particular model of a parity clash described here is its simplicity. On the other hand, although our interaction is only phenomenological, it bears no similarity to the usual leptonic weak interaction, which may be regarded by some as more than offsetting its advantages.

RIASSUNTO (*)

Faccio l'ipotesi che l'interazione debole non leptonica possa essere del tutto dissimile dalla interazione debole leptonica. Con questa ipotesi do un semplice modello della intergzione debole non leptonica. Il modello è basato sull'idea di Pais di un « conflitto di parità », ed è costruito in modo da dare la regola $|\Delta T| = \frac{1}{2}$ ed i parametri di asimmetria che si osservano nel decadimento pionico degli iperoni.

(*) *Traduzione a cura della Redazione.*

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Evaporation Mechanism and Direct Effect in n, p Reactions on Medium Weight Nuclei.

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Summary. — Energy spectra of protons emitted in (n, p) reactions with 14 MeV neutrons on 13 nuclides from ^{31}P to ^{103}Rh are presented. The presence of both evaporation and direct mechanisms is put in evidence. By comparing the forward with backward spectra an evaluation of direct effect cross-section is given.

1. – Introduction.

The problem of the mechanism of nuclear reactions at intermediate energies has aroused a great interest in the last few years: one of the more discussed points concerns the presence of two very different kinds of mechanisms acting in these reactions, namely, the statistical evaporation and the direct effect. Both of them have been shown to be active, but it is still a matter of research to establish how much of a particular reaction goes through each one of these mechanisms.

This problem is interesting in the case of n, p reactions. In a group of previous results ([1-3]), a few points have been established concerning the general behaviour of these reactions.

The first point is that the energy distribution of protons emitted in the forward direction in respect to the incident neutron beam is often in disagreement with the one foreseen by the evaporation theory; the second point is that a contribution of deuterons from (n, d) reactions has been found, in many cases, of the same order of magnitude of the n, p reaction. The necessity of a technique capable of discriminating between particles of different masses was therefore important.

After introducing this technique, the n, p reaction on rather light nuclei (^{24}Mg, ^{32}S, ^{28}Si) ([4,5]) has been studied in a number of cases.

Through angular distribution measurements, it was possible to show the existence of both mechanisms of evaporation and direct effect, and, moreover, to establish some properties of the direct effect. This was found to behave as a stripping-like surface direct effect, and to correspond, at least qualitatively, to the description given by BUTLER.

In order to extend this study to heavier nuclei, we have studied the proton spectrum emitted in the forward direction by a group of nuclei. The proton spectra from the thirteen elements ^{31}P, ^{40}Ca, ^{45}Sc, ^{51}V, ^{52}Cr, ^{55}Mn, ^{54}Fe, ^{56}Fe, ^{58}Ni, ^{60}Ni, ^{59}Co, ^{63}Cu, ^{103}Rh are presented in this paper.

In these nuclei it was soon evident that the evaporated part was much more important than in the lighter ones.

The properties of evaporation in n, p reactions at 14 MeV have been recently studied by FACCHINI *et al.* ([6]), in whose work there was found a general agreement between the evaporation theory and experimental energy spectra and cross-sections of protons emitted from n, p, and, due to that agreement, some fundamental characters of evaporation can be deduced, particularly the level density shapes.

At the same time, a group of nuclei in the same mass range was studied by ALLAN ([7]), who obtained energy spectra of protons emitted in the backward

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([1]) L. COLLI, U. FACCHINI, I. IORI, M. G. MARCAZZAN and A. M. SONA: *Nuovo Cimento*, **13**, 730 (1959).

([2]) L. COLLI, F. CVELBAR, S. MICHELETTI and M. PIGNANELLI: *Nuovo Cimento*, **13**, 868 (1959).

([3]) L. COLLI, S. MICHELETTI, M. PIGNANELLI and I. IORI: *Nuovo Cimento*, **20**, 94 (1961).

([4]) L. COLLI, I. IORI, M. G. MARCAZZAN, F. MERZARI, A. M. SONA and P. G. SONA: *Nuovo Cimento*, **17**, 634 (1960).

([5]) L. COLLI, M. G. MARCAZZAN, F. MERZARI, P. G. SONA and P. TOMAŠ: *Nuovo Cimento*, **20**, 928 (1961).

([6]) U FACCHINI, I. IORI and E. MENICHELLA: *Nuovo Cimento*, **16**, 1109 (1960).

([7]) D. L. ALLAN: *Nucl. Phys.*, **24**, 274 (1961).

direction. A good agreement with evaporation model was also found by ALLAN for his own results.

These facts allows us to use the hypothesis of the evaporation with some confidence and to analyse the forward spectra in terms of evaporation model and to obtain form this analysis information on the direct effects. The analysis of our results, together with those mentioned above, can give a rather wide information on the mechanism of the proton emission with 14 MeV neutrons.

2. – Experimental apparatus and results.

The esperimental apparatus is the same as described in the previous papers. The 14 MeV neutrons are obtained from the d – t reaction. The proton detector is a telescope of proportional and scintillation counters (8): mass discrimination of the emitted particles is used, so that our results are not contaminated by deuterons from n, d reactions. All the measurements have been taken at an angle of proton emission of 15° with an angular aperture of about 30°.

The spectra of protons obtained in this way are shown in Fig. 1 to 13.

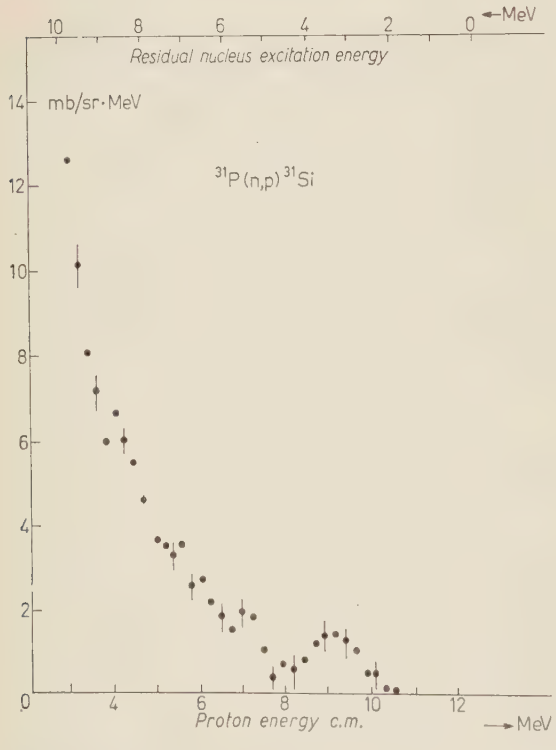

Fig. 1. – Proton energy spectrum from ^{31}P. The proton kinetic energy (lower scale) and residual nucleus excitation energy (upper scale) are shown on the abscissae. On the ordinates, the cross-section in mb/sr·MeV is shown.

(8) M. G. MARCAZZAN, M. PIGNANELLI and A. M. SONA: *Nuovo Cimento*, **10**, 155 (1958).

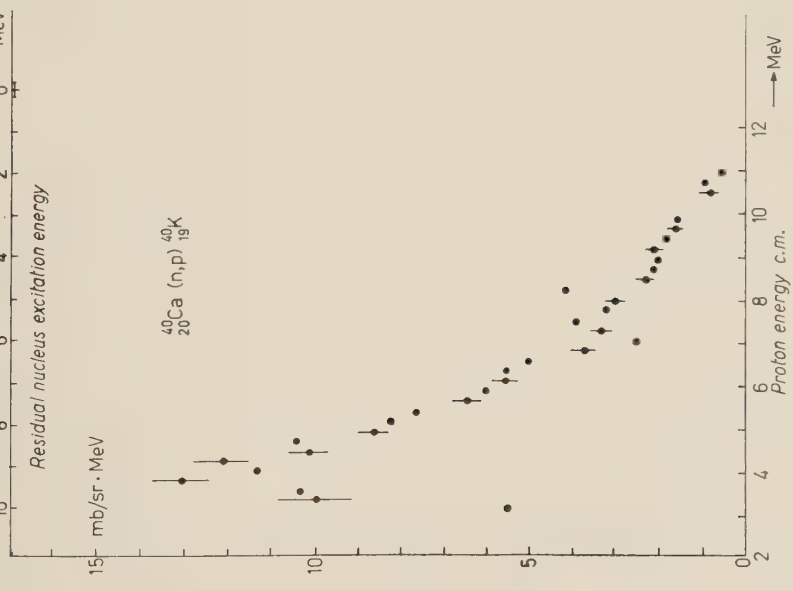

Fig. 2. – Proton energy spectrum from ^{40}Ca.

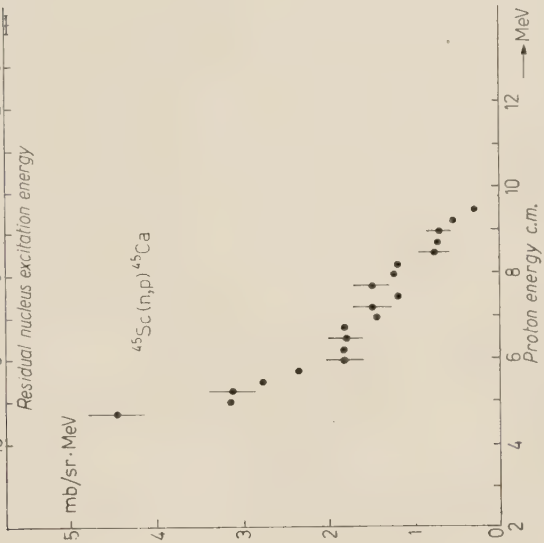

Fig. 3. – Proton energy spectrum from ^{45}Sc.

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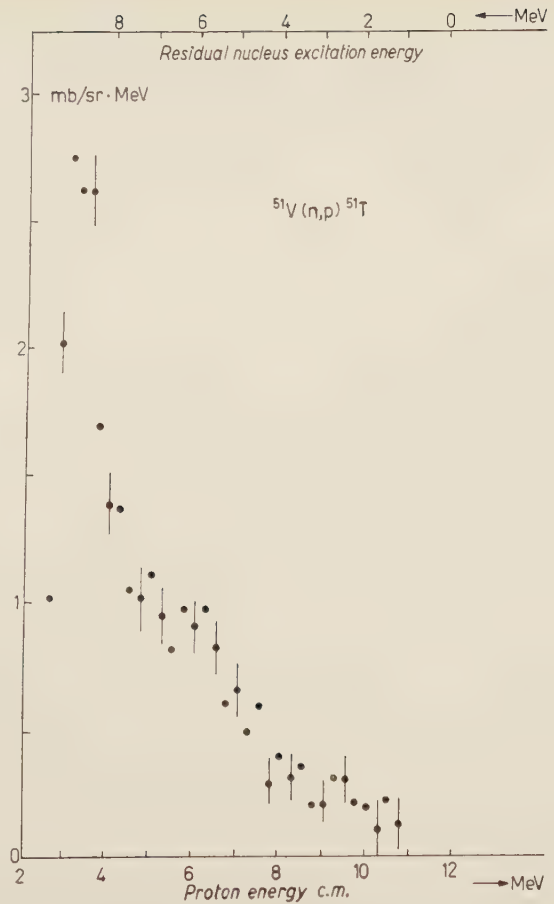

Fig. 4. – Proton energy spectrum from ^{51}V.

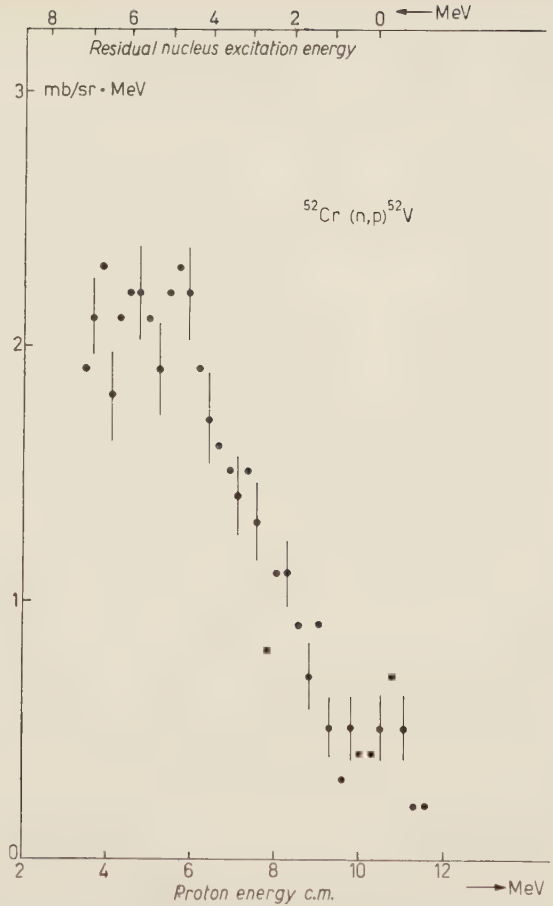

Fig. 5. – Proton energy spectrum from ^{52}Cr.

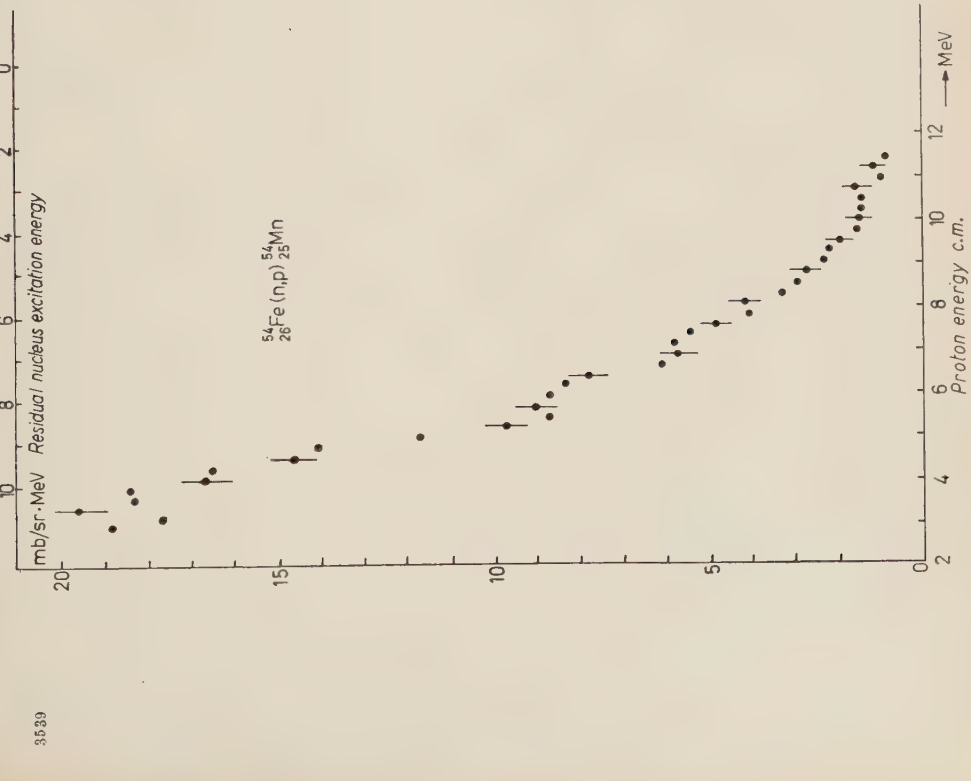

Fig. 6. – Proton energy spectrum from ^{54}Fe.

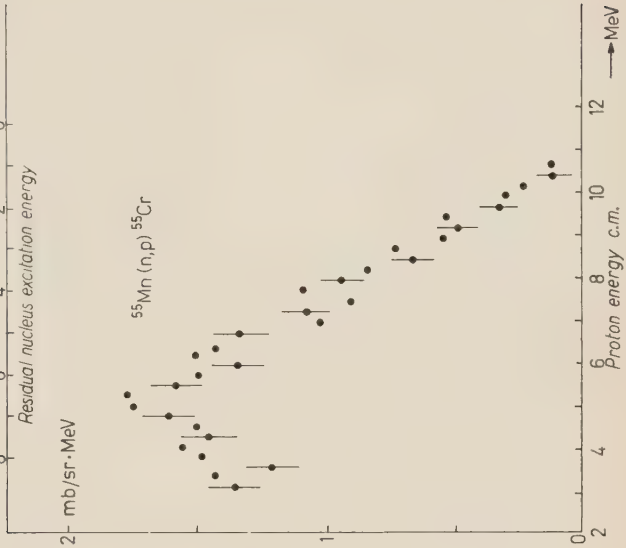

Fig. 7. – Proton energy spectrum from ^{55}Mn.

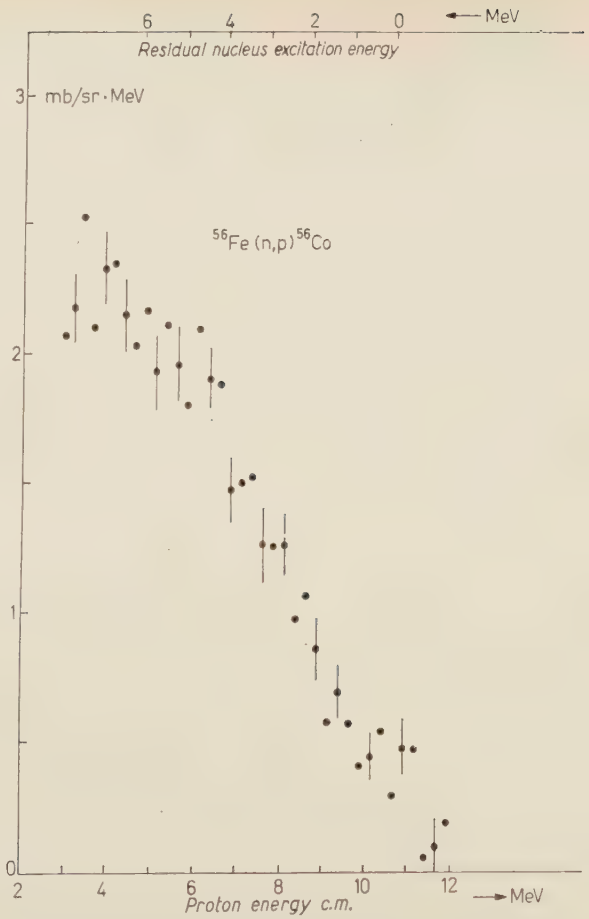

Fig. 8. – Proton energy spectrum from ^{56}Fe.

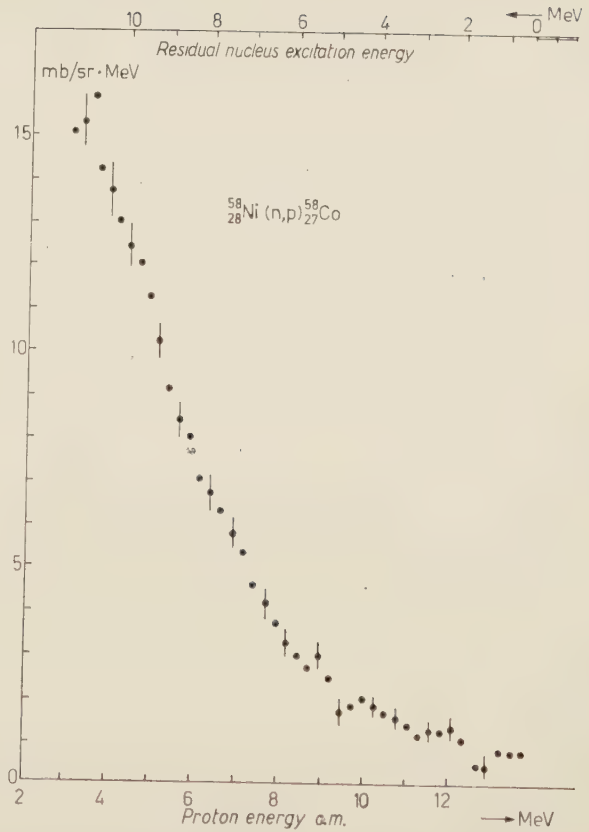

Fig. 9. – Proton energy spectrum from ^{58}Ni.

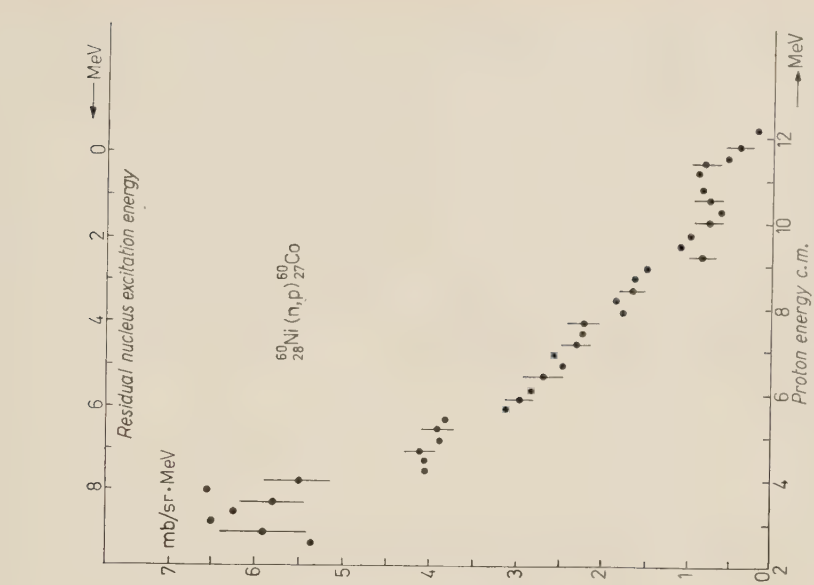

Fig. 11. – Proton energy spectrum from ^{60}Ni.

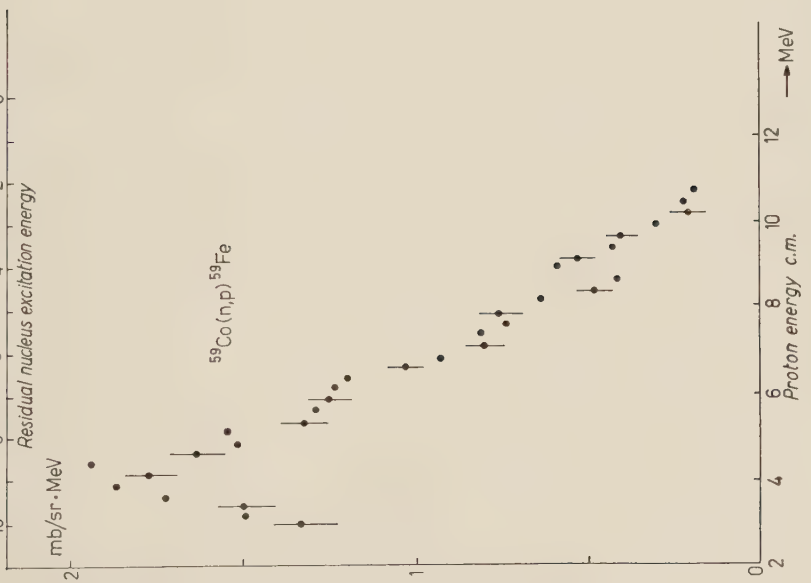

Fig. 10. – Proton energy spectrum from ^{59}Co.

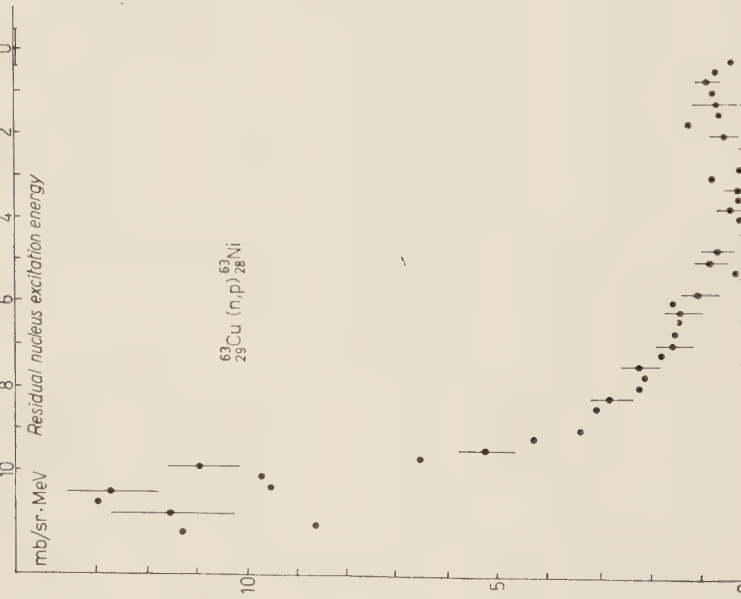

Fig. 12. – Proton energy spectrum from ^{63}Cu.

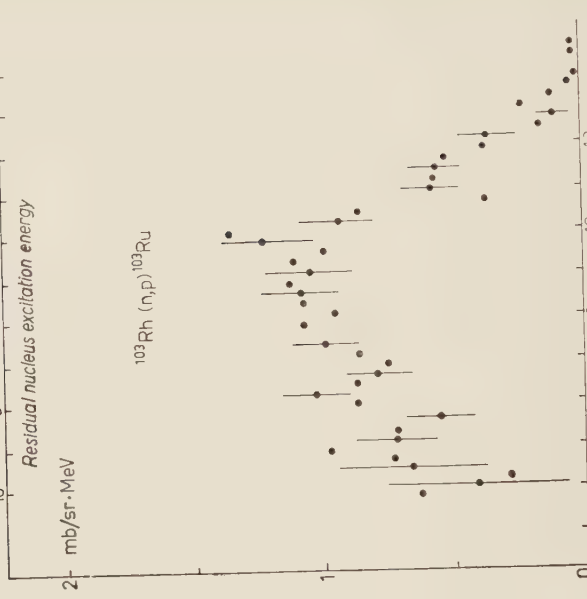

Fig. 13. – Proton energy spectrum from ^{103}Rh.

3. – Discussion of results.

An overall observation of the proton spectra puts in evidence two facts:

1) The spectrum shape is generally smooth with predominance of low energy protons, as expected from an evaporation process.

2) In a few cases some structures are shown (^{31}P, ^{40}Ca, ^{45}Sc, ^{51}V), which give evidence of the presence of direct effect in the reaction.

Therefore both mechanisms seem to be present.

In order to determine the relative quantity of them, we will firstly analyse the spectra from the point of view of the statistical evaporation theory.

Following the formula given by Weisskopf, the evaporation spectrum is given by

$$N(\varepsilon) \div \varepsilon\sigma_c\omega(\varepsilon_{\max} - \varepsilon)\,, \tag{1}$$

where

ε is the proton kinetic energy,
$\varepsilon_{\max}$ is the maximum allowed proton kinetic energy,
σ_c is the inverse process cross-section,
$\omega(\varepsilon_{\max} - \varepsilon)$ is the residual nucleus level density.

We shall put in the following $\varepsilon_{\max} - \varepsilon = E$, E being the residual nucleus excitation energy.

To gain information on nuclear properties we must extract from the spectrum the function $\omega(E)$, residual nucleus level density, that is, we must divide the spectrum by the quantity $\varepsilon\sigma_c$. We have taken σ_c from Weisskopf's calculations assuming $r_0 = 1.6A^{\frac{1}{3}}$; it has been shown (6) that the

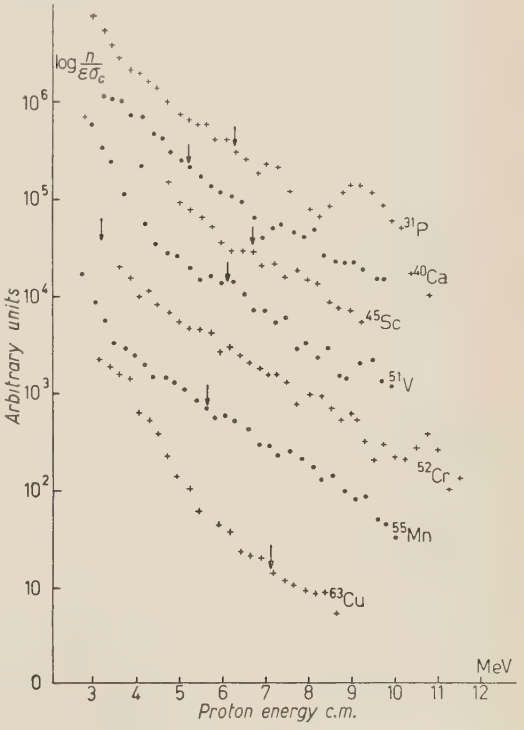

Fig. 14. – Forward proton spectra plotted as $n(\varepsilon)/\varepsilon\sigma_c$ for the nuclei ^{31}P, ^{40}Ca, ^{45}Sc, ^{51}V, ^{52}Cr, ^{55}Mn, ^{63}Cu. Ordinates are given in arbitrary units. Maximum energy of protons from (n, np) reactions are indicated with an arrow.

plotted the curves for various values of the scattering length, normalized to the same total area, at the incident energy of 624 MeV. Also shown are the curve obtained by the simple phase space, normalized to the same area as before. It is seen from the figure that it is impossible to observe experimentally that the maximum, in the « enhancing » S-wave case, does not correspond to the same produced mass, since the differences in the p_3 scale are too small to be appreciated. The form of the curve is however completely different. In this way one could discriminate the two possibilities. However we observe that:

peaks (forward and backward production) would mix their contributions in the p_3 spectrum.

Consequently better discrimination between the two possibilities could be obtained by a new experiment at 743 MeV but at another laboratory angle of the recoiling ^{3}He, as pointed out in ABASHIAN, BOOTH and CROWE ([6]).

For this aim we have done calculations at 15° laboratory angle which seems interesting since, in such condition, in the range of the values of p_3 reasonable for the experimental analysis, the momentum spectrum would show both the peaks of the single particle.

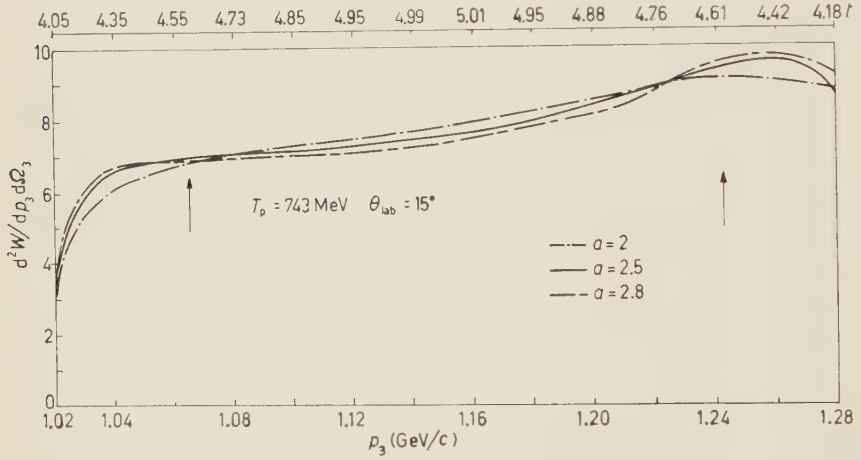

Fig. 3. – Momentum spectrum of ^{3}He at 15° laboratory system and 743 MeV proton incident energy. Curves normalized to equal area. Ordinate in arbitrary units.

1) At that incident energy (624 MeV) the momentum spectrum ranges on values of p_3 near to 1 GeV which makes the experiment very difficult (see ABASHIAN, BOOTH and CROWE ([6])).

2) Assuming the width of the particle as given by ABASHIAN, BOOTH and CROWE ([1]) at 624 MeV the two

Fig. 3 shows the yield of the S-wave model with different scattering length, at 743 MeV and 15° laboratory angle.

Observe that at this angle the variations of the cross-section due to different values of the scattering length are less relevant than at 12°. This fact should allow a clear cut distinction between this interpretation and the effects of the production of a particle of mass around 300 MeV which would give rise to peaks at the positions indicated in the Fig. 3 by the arrows.

([6]) A. ABASHIAN, N. E. BOOTH and K. M. CROWE: preprint May 1961.

In conclusion the S-wave model contains a parameter (a_0) which could be determined by means of the best fit of the experimental data at a fixed energy (743 MeV) and a fixed angle (12°). Such a value for the same incident energy but at another angle allows to construct up theoretically a ^{3}He momentum spectrum which, by comparison with the experimental one, should give an unambiguous answer on the validity of the proposed model.

* * *

I am deeply indebted to Dr. V. De Alfaro for help and extremely useful discussions.

I wish to thank warmly Prof. M. Cini for the useful discussions.

The analysis of FACCHINI and coworkers and of ALLAN show that in the backward region the direct effect (at least for big cross-sections) are practically unimportant. Moreover it has been shown (6) that the formula (2) is quite well approximated by an exponential law $\omega \sim \exp[E/\theta]$ over a range of values of E of few MeV.

The set of θ values obtained in some of our previous works (11) and in ALLAN's can resonably represent the evaporation behaviour.

The comparison treats two points:

1) cross-section values in the low energy region;

2) shape of the spectra.

1) The cross-sections in the low energy region agree with Allan's values for backward spectra, the limit of exponential curves showing that direct effects are not important in this energy region and giving a quite good reliability to the experimental results.

2) The comparison of shapes is simply obtained comparing the two sets of values: the temperatures calculated from backward results are always smaller than the ones obtained from our results. So it is evident that a component of protons at the highest energies is present in the forward spectra.

It seems natural to assume that, in the spectra presented here, the structures and deviations from evaporation shape found at the highest proton energies must be attributed to the presence of direct effect.

Therefore, following this line of thinking, we can assume that the direct effect present in the nuclei discussed here have similar properties to the one we have found in the three cases quoted above, that is, we expect an angular distribution as given by surface stripping-like interaction.

This angular distribution cannot be foreseen exactly, because we can only guess which proton and neutron shell model orbits take part in the reaction, but we can at least assume that the effect will be predominantly in the forward hemisphere, since the contribution from high values of the transferred angular momenta is certainly small.

In order to find the direct effect contribution in the proton spectra taken in forward direction, we must subtract the cross-section value for the backward emission from the forward one.

But, the direct effect contribution being quite small compared to the main evaporated part of the proton spectrum, the results so obtained would have such a big error as to be practically without meaning.

(11) L. COLLI, U. FACCHINI, I. IORI, M. G. MARCAZZAN, A. M. SONA and M. PIGNANELLI: *Nuovo Cimento*, **7**, 400 (1958); L. COLLI, U. FACCHINI, I. IORI, M. G. MARCAZZAN and A. M. SONA: *Nuovo Cimento*, **13**, 730 (1959).

Because of these absolute errors, we have decided to use only the shape of Allan's spectra emitted in the backward direction: that is, we have normalized them to our spectra using as normalization criterion the equality of the corresponding spectra in the proton energy range from 3 to 5 MeV, and then we have integrated the two spectra on the energy from 5 MeV to maximum proton energy.

The difference between the two integrals gives the amount of direct effect in forward direction; the underlying hypothesis is that in the $(3 \div 5)$ MeV energy range there is no contribution from direct effect. The results of this calculation are shown in Table I, last column.

Comparing these direct effect cross-section values with the one obtained in the case of Si, Mg and S, we find that they are of the same order of magnitude. This can be an indication of the fact that in the mass range considered here the direct effect is more or less the same as for lighter masses.

* * *

We cordially thank Dr. D. L. Allan for having kindly shown us his results before publication, Prof. U. Facchini for some useful discussions and F. Cvelbar for his participation in a part of the measurements.

RIASSUNTO

Sono presentati gli spettri di energia dei protoni emessi da 13 nuclei compresi tra il ^{31}P e il ^{103}Rh. Viene mostrata l'esistenza dei due meccanismi di reazione: l'evaporazione statistica e l'effetto diretto. Confrontando la forma degli spettri dei protoni emessi in avanti con quelli emessi all'indietro si ottiene una valutazione della sezione d'urto dell'effetto diretto.

The Electric Field in Single-Turn and Multi-Sector Coils.

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(ricevuto il 13 Giugno 1961)

Summary. — Calculations are reported of the distribution of electric field within long single-turn and multi-sector coils. Some of the calculations refer to coils containing a cylindrical conductor, the latter representing an idealized plasma.

1. – Introduction.

In the present paper calculations are presented which refer to the electric field configurations within long single-turn coils and also within multi-sector coils (1). Such coils are used at present in plasma physics research, both in experimental work on the orthogonal (or theta) pinch, and also sometimes in experiments with radio-frequency discharges.

The first calculations refer to the electric field distributions which exist in the absence of plasma, *i.e.* the distributions which exist before the electrical breakdown of the gas.

Further calculations refer to a coil which contains a cylindrical conductor. Two cases are considered, namely: the case of a rigid conductor and the case of a conductor which is collapsing towards the axis. The first case may approximate to that existing in certain radio-frequency experiments while the second represents an idealized orthogonal pinch experiment.

It is assumed that the frequency is low enough for the displacement currents to be negligible. This assumption is valid if the radius of the coil is small compared with the wavelength of the oscillation.

(1) J. E. Allen and S. E. Segre: *Proceedings of the IV International Conference on Ionization Phenomena in Gases, Uppsala 1959* (Amsterdam, 1960), p. 1073.

2. – The resolution of an electric field into « electrostatic » and « inductive » components.

A vector field can always be uniquely resolved into two components one being irrotational (curl-free), and the other solenoidal (divergence-free), if the field vanishes at infinity. Thus an electric field can be split into two components

$$\boldsymbol{E} = \boldsymbol{E}_1 + \boldsymbol{E}_2 , \tag{1}$$

where $\operatorname{curl} \boldsymbol{E}_1 = 0$ and $\operatorname{div} \boldsymbol{E}_2 = 0$.

The electric field must satisfy the Maxwell equation

$$\operatorname{curl} \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{H}}{\partial t} ,$$

and therefore it may be written in the form

$$\boldsymbol{E} = -\operatorname{grad} \Phi - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} , \tag{2}$$

where Φ is a scalar potential and $\boldsymbol{A}$ is a vector potential defined by the relation

$$\boldsymbol{H} = \operatorname{curl} \boldsymbol{A} .$$

The vector potential is not completely defined by its curl but its divergence must also be specified. If $\operatorname{div} \boldsymbol{A}$ is arbitrarily chosen to be zero, then the two components of eq. (1) can be identified with those of eq. (2). These two components may be referred to as the « electrostatic » field and the « induced » electric field respectively.

The choice of $\operatorname{div} \boldsymbol{A} = 0$ is known as the Coulomb gauge because the scalar potential in eq. (2) is simply that corresponding to charges at rest, namely $\nabla^2 \Phi = 4\pi\varrho$, since $\operatorname{div} \boldsymbol{E} = \operatorname{div} \boldsymbol{E}_1 = 4\pi\varrho$. Thus the use of the term « electrostatic » field for $\boldsymbol{E}_1$ is not without justification.

The division of $\boldsymbol{E}$ into these components can be made whether the displacement current is negligible or not. When the latter is not negligible, however, a different gauge is sometimes more convenient.

In some theories a uniform field exists at infinity; it is purely arbitrary whether this field is derived from a vector or from a scalar potential.

3. – The electric field in a single-turn coil.

Cylindrical co-ordinates, r, θ, z, will be used throughout the paper and it will always be assumed that $\partial/\partial z = 0$.

A single-turn coil is now considered in which: *a*) the length is large compared with the radius R so that the magnetic field is in the z direction and is constant in space, *b*) the width of the gap in the coil is very small compared with R, and *c*) the conductivity of the coil is so high that $E_\theta(R)$ is vanishingly small.

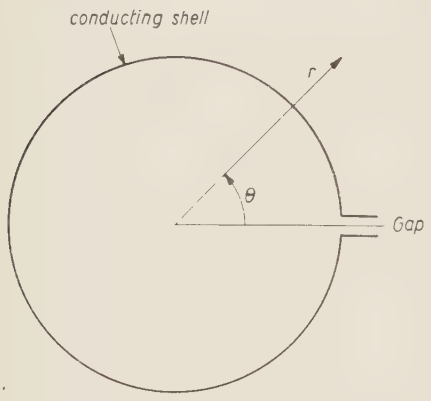

Fig. 1. – Diagram of the single-turn coil.

As the currents are purely azimuthal and independent of θ, $\boldsymbol{A}$ is in the θ direction and so $\boldsymbol{E}_2$ is also in this direction. If a voltage V_0 is applied to the gap (*), (which is at $r = R$, $\theta = 0$, as shown in Fig. 1), then the components of the induced electric field $\boldsymbol{E}_2$, are given by

$$E_{2\theta} = -\frac{V_0}{2\pi R}\,\rho\,, \qquad E_{2r} = 0\,,$$

where $r/R = \rho$. The electrostatic field $\boldsymbol{E}_1$ must satisfy the conditions that $\operatorname{curl} \boldsymbol{E}_1 = 0$, $\operatorname{div} \boldsymbol{E}_1 = 0$ for $\rho < 1$ and $E_{1\theta} + E_{2\theta} = 0$ for $\rho = 1$. These conditions are satisfied by the following field:

$$E_{1\theta} = \frac{V_0}{\pi R}\,\frac{(\cos\theta - \rho)}{(1 + \rho^2 - 2\rho\cos\theta)}\,, \qquad E_{1r} = \frac{V_0}{\pi R}\,\frac{\sin\theta}{(1 + \rho^2 - 2\rho\cos\theta)}\,.$$

This is a Biot-Savart field whose axis is on the gap, and the equipotentials are planes passing through the gap. The surface charge density on the coil (where $\rho = 1$) is proportional to E_{1r}, *i.e.* to $[V_0/2\pi R]\operatorname{ctg}\theta/2$.

The field $\boldsymbol{E}_1$ is conveniently represented by the complex function $f(z, \rho) = (V_0/\pi R)/(z - \rho)$ whose real and imaginary parts are respectively $E_{1\theta}$ and E_{1r}, if $z = \exp[-i\theta]$.

The lines of force of the total field $\boldsymbol{E}$ are obtained by integrating the following differential equation

$$\frac{E_\theta}{E_r} = \rho\,\frac{\mathrm{d}\theta}{\mathrm{d}\rho}\,. \tag{3}$$

(*) The sign of V_0 is defined so that $V_0 = -\pi R^2 \dot{H}_z/c$.

Putting $\cos\theta = \mu$

$$\frac{\mathrm{d}\mu}{\mathrm{d}p} = \frac{1}{2}\, p^{-1}(1-p^2)(p-2\mu)\,,$$

and integration then gives the following equation for the lines of force

$$\mu = p^{-1}\left[\frac{(1+p^2)}{2} - (1-\mu_0)\cdot \exp\left[-\frac{(1-p^2)}{2}\right]\right],$$

where μ_0, the constant of integration, is the value of μ at $p=1$ and is the parameter of the family of curves representing the lines of force (see Fig. 2).

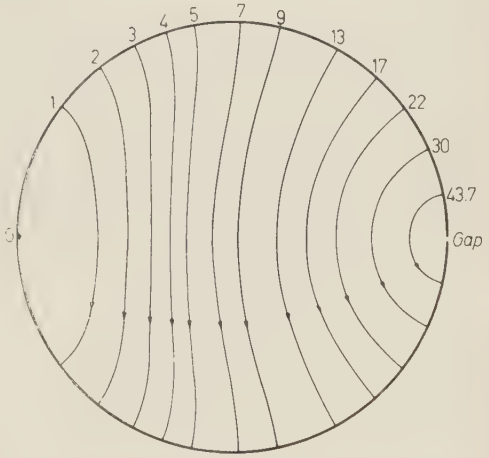

Fig. 2. - The lines of force of the electric field inside a single-turn coil. The numbers are the values of the flux function in arbitrary units.

An electric field exists not only inside but also outside the single-turn coil described above, and it can easily be seen that, for the external field, $\boldsymbol{E}_2$ is a Biot-Savart field of unit strength whose axis is at $p=0$, while $\boldsymbol{E}_1$ results from the sum of two Biot-Savart fields, one of strength -2 with its axis at $p=0$, and the other of strength $+2$ with its axis on the gap. A field of unit strength, in this context, has a circulation equal to V_0.

4. – The electric field in a multisector coil with n gaps.

The multisector coil [1] consists of a cylindrical conducting shell which is divided by a certain number (n) of gaps (parallel to the axis), so that there are n conducting sectors. A voltage V_0 is applied across each gap and the same assumptions are made as in the preceding paragraph with reference to the length and conductivity of the coil and to the width of the gaps.

The problem is linear and therefore the contributions of each gap may simply be added in order to obtain the total field. The total induced electric field is therefore given by

$$E_{z\theta} = \frac{nV_0}{2\pi R}\, p\,, \qquad E_{zr} = 0\,.$$

The total electrostatic field may be obtained by the use of the function $f(z, p)$ defined above. If the position of the k-th gap is at $p = 1$, $\theta = \alpha_k$, $(k = 1, 2, \ldots n)$, then its contribution to $\boldsymbol{E}_1$ may be represented by

$$f(z_k, p) = \frac{V_0}{\pi R} \frac{1}{(z_k - p)},$$

where $z_k = z \exp[i\alpha_k]$ and the total $\boldsymbol{E}_1$ is represented by

$$(4) \qquad \sum_{k=1}^{n} f(z_k, p) = \frac{V_0}{\pi R} \sum_{k=1}^{n} \frac{1}{(z_k - p)} = -\frac{V_0}{\pi R} \frac{\mathrm{d}}{\mathrm{d}p} \left[\ln \prod_{k=1}^{n} (z_k - p) \right].$$

If the gaps are equally spaced, $\alpha_k = 2\pi k/n$ and $z_k^n = z^n$. Therefore the z_k's are the roots of the equation $w^n - z^n = 0$, and $\prod_{k=1}^{n} (w - z_k) = w^n - z^n$.

Hence $\prod_{k=1}^{n} (z_k - p) = (-1)^n (p^n - z^n)$

and $\sum_{k=1}^{n} f(z_k, p) = -\frac{V_0}{\pi R} \frac{np^{(n-1)}}{(p^n - z^n)}$.

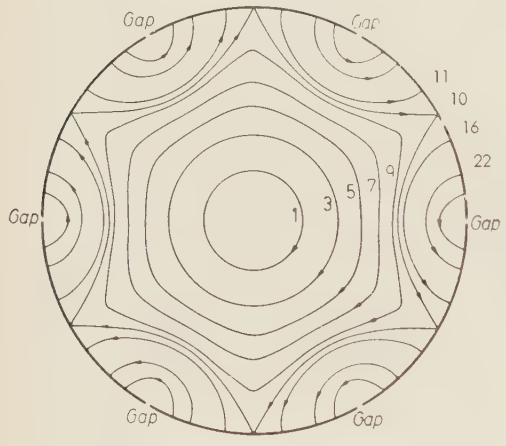

Fig. 3. – The lines of force of the electric field inside a six-sector coil. The numbers are the values of the flux function in arbitrary units.

The components $E_{1\theta}$ and E_{1r} are the real and imaginary parts of this function, *i.e.*

$$(5) \qquad E_{1\theta} = = \frac{nV_0}{\pi R} \frac{p^{n-1}(\cos n\theta - p^n)}{(1 + p^{2n} - 2p^n \cos n\theta)},$$

$$(6) \qquad E_{1r} = = \frac{nV_0}{\pi R} \frac{p^{(n-1)} \sin n\theta}{(1 + p^{2n} - 2p^n \cos n\theta)}.$$

The lines of force for the total field are obtained by integrating eq. (3), which now has the form

$$-\frac{\mathrm{d}\mu}{\mathrm{d}p} = \frac{n}{2} p^{-n} [p^{(2n+1)} - 2p^{(2n-1)} + p - 2\mu(p^2 - 1)p^{(n-1)}],$$

where $\mu = \cos n\theta$. Integration gives the following equation for the lines of force

$$\mu = p^{-n} \left[\frac{(1 + p^n)}{2} - (1 - \mu_0) \exp\left[-\frac{n(1 - p^2)}{2} \right] \right].$$

Fig. 3 shows the lines of force for the case where $n = 6$. It may be seen from eq. (5) and (6) that $E_1 \to 0$ as $n \to \infty$ (for $p < 1$).

It should be noted that eq. (4) is general in that it also applies to the case where the α_k's are not integral multiples of $2\pi/n$ and it can be extended immediately to include the case when a voltage V_k is applied to the k-th gap.

5. – The electric field in a single-turn coil containing a conducting core.

The next case to be considered is that of a single-turn coil of radius R containing a cylindrical conducting core, of radius $r_0 = p_0 R$, which is coaxial with the coil. The same assumptions for the coil will be made as in Section **3** and the core will be supposed to be of high conductivity. The gap, as shown in Fig. 4, is at $p = 1$, $\theta = \pi$.

Two cases will be considered namely: *a*) the case where the core is a rigid conductor, and *b*) the case in which the current in the coil is constant and $r_0 = r_0(t)$. In both cases the induced electric field is purely azimuthal, as in Section **3**, and it can easily be shown that, in case *a*),

$$E_{2\theta} = \frac{V_0}{2\pi R} \frac{1}{p} \cdot \frac{(p^2 - p_0^2)}{(1 - p_0^2)}, \qquad E_{2r} = 0,$$

where V_0 is the voltage applied to the gap, and in case *b*)

$$E_{2\theta} = \frac{4\pi I \dot{r}_0}{c^2} \frac{p_0}{p}, \qquad E_{2r} = 0,$$

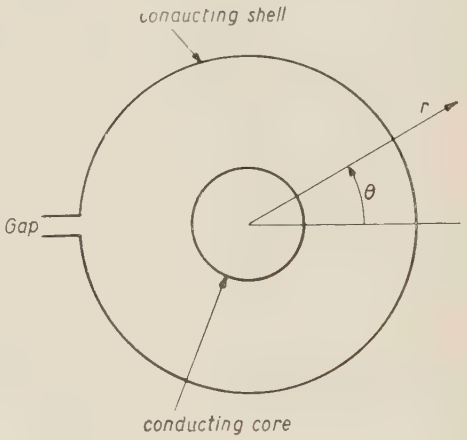

Fig. 4. – Diagram of a single-turn coil with a coaxial conductor.

where I is the (constant) current per unit length of coil.

In both cases $E_{1\theta}$ must be a constant for $p = 1$, (for $p = 1$, $E_{1\theta} = -V_0/(2\pi R)$ in case *a*) and $E_{1\theta} = -4\pi I \dot{r}_0 p_0 c^{-2}$ in case *b*)) and in both cases $E_{1\theta}$ must be zero for $p = p_0$, (for $p = p_0$, $E_\theta = 0$ in case *a*) and $E_\theta = 4\pi I \dot{r}_0 c^{-2}$ in case *b*)). Therefore the electrostatic part of the field is the same in both cases. It will be determined by use of the electrostatic potential $\Phi(p, \theta)$.

The potential must satisfy Laplace's equation for $p_0 < p < 1$ and the following boundary conditions: $\Phi(p_0, \theta) = 0$ and $\Phi(1, \theta) = K\theta$, where $K = V_0/(2\pi)$ in case *a*) and $K = 4\pi I \dot{r}_0 p_0 R c^{-2}$ in case *b*).

The general solution of the equation of Laplace in two dimensions can be

written in the following form

$$\Phi(p, \theta) = A \ln p + \sum_{n=-\infty}^{\infty} p^n (A_n \cos n\theta + B_n \sin n\theta) .$$

Using the boundary conditions it is possible to determine the constants A, A_n and B_n and so obtain

$$\Phi(p, \theta) = -\sum_{n=1}^{\infty} 2K \frac{\sin n\theta}{n} \frac{(-p)^n}{(1 - p_0^{2n})} \left[1 - \left(\frac{p_0}{p}\right)^{2n}\right] .$$

The electrostatic field is given by the gradient of this function, *i.e.*

$$E_{1\theta} = -\frac{1}{pK} \frac{\partial \Phi}{\partial \theta} = \frac{2K}{pR} \sum_{n=1}^{\infty} \frac{\cos n\theta (-p)^n}{(1 - p_0^{2n})} \left[1 - \left(\frac{p_0}{p}\right)^{2n}\right] .$$

$$E_{1r} = -\frac{1}{K} \frac{\partial \Phi}{\partial p} = \frac{2K}{pK} \sum_{n=1}^{\infty} \frac{\sin n\theta (-p)^n}{(1 - p_0^{2n})} \left[1 + \left(\frac{p_0}{p}\right)^{2n}\right] .$$

The differential eq. (3) for the lines of force has been integrated numerically

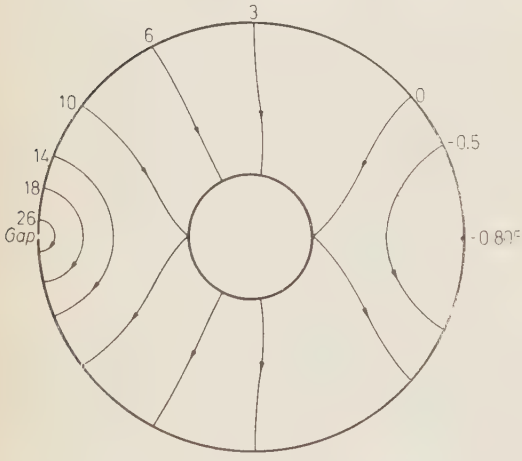

Fig. 5. – The lines of force of the electric field in a single-turn coil with a rigid conducting core, (case *a*). The numbers are the values of the flux function in arbitrary units and the ratio of the radius of the core to that of the coil is 0.3.

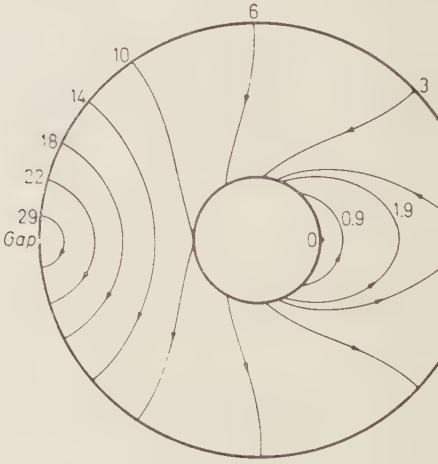

Fig. 6. – The lines of force of the electric field in a single-turn coil which carries a constant current and contains a collapsing (or expanding) conducting core (case *b*). The numbers are valuesof the flux function in arbitrary units and the ratio of the radius of the core to that of the coil is 0.3

both for case *a*) and case *b*) (for $p_0 = 0.3$) and plots of the lines of force are shown in Fig. 5 and Fig. 6.

6. – Discussion.

Electric fields such as those discussed in Sections **3** and **4** occur in plasma devices [1] before breakdown and they are important in determining the development of the breakdown (*). Some theoretical studies [2] of breakdown consider only the induced electric field $\boldsymbol{E}_2$ and not the electrostatic field $\boldsymbol{E}_1$. It should be noted (Fig. 2) that in a single-turn coil there are no closed lines of force but all the lines of force end on the conductor and the intensity of the total electric field is largely determined by the electrostatic component. With a multi-sector coil an approximately azimuthal field is obtained near the axis, as shown in Fig. 3.

The electric fields discussed in Section **5** are important in connection with experiments in which a highly conducting column of plasma has been formed (*). The electric field in an orthogonal pinch device will in general result from the sum of the fields of case *a*) and case *b*), the former corresponding to the voltage due to change of current and the latter to the voltage due to change of inductance. The field is more nearly that of case *a*) alone in stationary R.F. experiments (or in orthogonal pinch experiments at times when $\dot{i}_0 = 0$). The electrostatic field vanishes in a shortcircuited coil but the induced field is not necessarily zero.

The electric field acts on the tenuous, weakly-conducting gas outside the plasma, where it may produce further ionization and also runaway electrons. The electric field should be considered, as a boundary condition, in calculations referring to the sheath at the surface of the plasma.

* * *

The authors wish to thank Prof. B. Brunelli for discussions and for his interest in this work. They are also indebted to Dr. A. Turrin for the numerical integrations.

(*) The supply may be connected to a coil which is outside the conductors shown in Figs. 1 and 4, or the multi-sector structure of Section **4**, but the distribution of the internal electric field remains unaltered.

(2) H. U. Eckert: *Proceedings of the IV International Conference on Ionization Phenomena in Gases, Uppsala* 1959, (Amsterdam, 1960), p. 320.

RIASSUNTO

Si riferisce su calcoli della distribuzione di campo elettrico in avvolgimenti lunghi a uno o più settori. Alcuni dei calcoli si riferiscono ad avvolgimenti contenenti un conduttore cilindrico che rappresenta un plasma idealizzato.

Some Analytic Properties of a Decay Amplitude with Final State Interactions.

II - The Single-Loop Four-Point Diagrams in K $\to 3\pi$ Decay (*).

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(ricevuto il 17 Giugno 1961)

Summary. — We investigate the analytic properties of the simplest single loop Feynman diagrams whose contributions to $K \to 3\pi$ decay depend on both of the independent energy variables. This is done by replacing the two internal double pion lines by fictitious single lines of discrete rest masses λ_1 and λ_2 and integrating over λ_1 and λ_2 at the end of the calculation. The analytic properties of the actual amplitude are those of the fictitious amplitude in the limit as λ_1 and λ_2 independently tend to a value of two pion masses from above. The amplitude has complex singularities and does not satisfy a Mandelstam-type representation. However, a one dimensional dispersion representation with only real cuts does hold in one variable when the other variable is fixed at any real value. We comment briefly on the change in the properties of the amplitude as the K-meson mass increases past the instability point.

1. – Introduction.

Our aim is to investigate the analytic properties of the simplest Feynman diagrams whose contributions to $K \to 3\pi$ decay depend on two of the cova-

(*) Work supported in part by the U. S. Atomic Energy Commission.

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(*.*) On leave of absence from Magdalen College, Oxford. Supported in part by the U.S.A.F. Office of Scientific Research, Air Research and Development Command.

riant variables, which are conveniently taken to be the negative squared sums of the momenta of the different pairings of pions. In an earlier paper (1), to be called I in the following, we considered the leading diagrams whose contributions depend each on only one of these variables; we refer the reader to I for a discussion of our approach to perturbation theory, to the problem of divergences (*), and to the treatment of the unstable particle. Here, as in I, we ignore the isotopic structure. Our study answers some of the questions raised by OKUN (2) about the validity of certain previous conjectures (3,4) about these two-variable dependent components of the amplitude.

The leading diagrams to depend on two of the variables are of order Gg^3, with G the weak and g the pion-pion coupling constant. They fall into the two types shown in Fig. 1; we shall consider only the single loop diagrams, *i.e.* only those of type (*a*). The heavy line denotes the incoming K-meson; all other lines represent pions. Let us for definiteness label the outgoing pion lines by their momenta as in Fig. 1 (*a*). Then the covariant variables natural to the diagram are the negative squared sums of adjacent momenta, *i.e.*

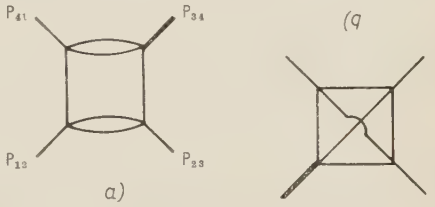

Fig. 1. The leading two-s-dependent diagrams.

$$-(p_{12}+p_{23})^2 = s\,,$$

and

$$-(p_{41}+p_{12})^2 = t\,.$$

In the following we shall, to begin with, confine ourselves to determining the analytic properties of this particular diagram regarded as a function of s and t. By permuting the external momenta amongst the external lines, a total of six diagrams is generated; when the contribution of each is written

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(1) G. BARTON and C. KACSER: *Nuovo Cimento*, **21**, 593 (1961).

(*) We stated in I that our diagrams were in principle renormalizable by analogy with the standard φ^4 theory. Actually the φ^4 theory is renormalizable only in the conventional sense of requiring only a finite number of counter-terms; there is known no unambiguous method for removing in practice the divergences from any given diagram.

(2) L. B. OKUN: *Proc. of the 1960 Rochester Conference*, p. 743.

(3) R. F. SAWYER and K. C. WALI: *Phys. Rev.*, **119**, 1429 (1960).

(4) S. FUBINI and R. STROFFOLINI: *Nuovo Cimento*, **17**, 263 (1960).

down in terms of the variables natural to it, they sum to a function that is formally symmetric in the three Mandelstam-type variables s, t and $u = -(p_{41}+p_{23})^2$. These variables of course satisfy the constraint

$$s+t+u = M^2+3\,, \tag{1.1}$$

where, as in the following, the pion mass is taken as unity, and M is the mass of the K-meson. We describe s, t, and u collectively as the s-variables, to distinguish them from the y-variables to be introduced below.

In the same way as in I, we replace the two internal double pion lines by single lines of rest masses λ_1 and λ_2 respectively. Denoting by f the amplitude calculated for this modified diagram, the actual amplitude is given by

$$F = \int_4^\infty \mathrm{d}\lambda_1^2 \sigma(\lambda_1^2) \int_4^\infty \mathrm{d}\lambda_2^2 \sigma(\lambda_2^2) f\,. \tag{1.2}$$

The only relevant singularity of $\sigma(\lambda^2)$ is its branch point at $\lambda^2 = 4$. Since the Feynman amplitude involves λ_1 only once, in the combination $(q^2+\lambda_1^2)^{-1}$, where q is the momentum of an internal line, it is clear that in the λ_1^2 plane f cannot have coincident singularities; therefore nothing prevents a deformation of the λ_1^2 contour, and the singularities of F are those of f as λ_1 tends to 2 from above. An analogous conclusion holds for λ_2.

The limiting case $\lambda_1 = 2 = \lambda_2$ is actually a degenerate point where a transition takes place between regions of different analytic properties; hence it will be necessary to consider f with λ_1 and λ_2 fixed at values above but arbitrarily close to 2. For simplicity we shall also take $\lambda_1 = \lambda_2 = \lambda$. This entails no loss of generality as long as we remember to lift the equality when we come to evaluate F from f by means of (1.2); (cf. Section 2.7 below).

2. – Determination of singularities.

2·1. *Variables.* – We adopt the y-variables introduced by KARPLUS, SOMMERFIELD, and WICHMANN [5]. Define

$$y_{12} = y_{23} = y_{41} = \lambda/2\,; \qquad y_{34} = -(M^2-\lambda^2-1)/2\lambda\,;$$

$$y_{13} = -(s-2)/2\,; \qquad y_{24} = -(t-2\lambda^2)/2\lambda^2\,;$$

[5] R. KARPLUS, C. M. SOMMERFIELD and E. WICHMANN: *Phys. Rev.*, **114**, 376 (1959).

also let $y_{11} = y_{22} = y_{33} = y_{44} = +1$. Note that y_{13} and y_{24} are related linearly, but with a sign change, to s and t respectively, and that in terms of the y-variables normal thresholds always occur at $y = -1$. It is worth keeping in mind that while the normal threshold for our particular diagram is at $t = 16$, the actual threshold of the full amplitude, of course, lies at $t = 4$. Note also the combination of external instability at one vertex ($y_{34} < -1$) with internal instability at all the other three vertices (y_{12}, y_{23}, $y_{41} > +1$) which was pointed out in I as characteristic of a decay process with final state interactions.

In terms of our variables the amplitude f, apart from factors not affecting its analytic properties, is given by [6]

$$f(y_{13}, y_{24}, y_{34}) = \int_0^1 \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \mathrm{d}\alpha_3 \mathrm{d}\alpha_4 \, \frac{\delta(1 - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4)}{\lambda_1 \lambda_2 D^2} ,$$

where

$$D = \Big\{ \sum_{i,j=1}^{4} \alpha_i y_{ij} \alpha_j - i\varepsilon \Big\} .$$

2·2. *Determination of singularities in the real y_{24}-y_{13} plane.* – In incorporating the Feynman $i\varepsilon$ into the y-variables and reinterpreting f as a function of complex variables we follow standard methods (cf. for instance references [6] and [8]) and will not enter into details. We confine ourselves throughout to the amplitude on the physical sheet, and begin by investigating its singularities in the real y_{24}-y_{13} plane, with y_{34} fixed at its physical value; as shown in I, this procedure can be made meaningful by assigning to y_{34} an infinitesimal negative imaginary part, corresponding to a positive imaginary part of M^2. (This amounts merely to ensuring that M^2 is on the physical side of its cut; cf. the discussion below.)

The singularities of f are confined to the algebraic curves obtained by setting equal to zero the determinant of the matrix y_{ij} or any of its principal minors ([6,8]). At present we are interested only in the real sections of these curves. The 2×2 minors yield only the lines of normal threshold singularities $y_{24} = -1$ and $y_{13} = -1$. The associated branch cuts run to $-\infty$. (Note that by setting $\lambda_1 = \lambda_2$ we have made the configuration in the y_{24}-y_{13} plane symmetric under interchange of y_{24} and y_{13}; this circumstance simplifies the discussion, and is to be kept in mind throughout the following.)

([6]) J. TARSKI: *Journ. Math. Phys.*, **1**, 149 (1960).

([7]) L. F. COOK and J. TARSKI: *Some properties of the five-point function in perturbation theory*, preprint (1961). To appear in *Journ. Math. Phys.*

([8]) R. J. EDEN: *Lectures on the use of perturbation methods in dispersion theory*, University of Maryland, Physics Department, Technical Report no. 211 (1961).

The four 3×3 minors lead to the so-called vertex singularities, which are the leading singularities of diagrams obtained from that of Fig. 1 (a) by short circuiting one of its internal lines. These four diagrams are shown in Fig. 2.

It follows immediately from the discussion in I (and can be checked trivially) that diagrams (c) and (d) possess only the normal threshold singularities. These are the diagrams whose contributions are independent of M. Diagram (a) was investigated in detail in I; we shall review its properties below. Diagram (b), expressed in terms of the y-variables, is obtained from diagram (a) merely by interchanging y_{24} and y_{13}.

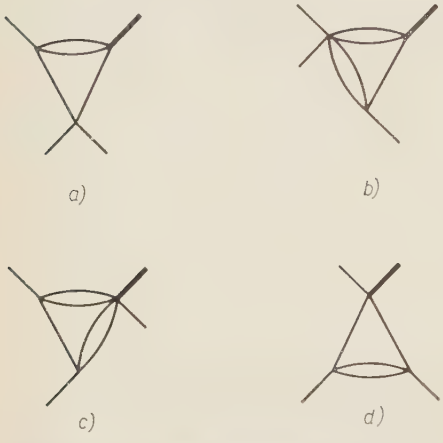


Fig. 2. The vertex contractions of Fig. 1a.

We now translate the conclusions of I into the language of the y-variables. Let us call the lower and upper edges of the normal branch cuts the physical and unphysical sides, respectively, to remind ourselves that the physical amplitude is obtained by letting y_{24}, y_{13}, and y_{34} tend to their real physical values from the lower side. With the understanding that y_{34} is fixed at its actual value on the physical side of its cut, we found in I that diagram 2 (a) has two vertex singularities in y_{13}, both lying on the normal branch cut. The more negative one, located at $y_{13} - L_1$, say, always lies on the unphysical edge; the other one, at $y_{13} = L_2$, say, can lie on either side, depending on the relative magnitudes of M and λ. For M fixed at its actual value and λ sufficiently close (and above) 2, both lie on the unphysical side. (In I, where we concentrated on the physical edge of the cut, singularities on the unphysical edge were not fully commented on.)

The leading singularities of diagram 1 (a), the so-called scattering singularities, lie on the curve

$$\det\{y_{ij}\} = 0 = (y_{13}-1)(y_{24}-1)\{(y_{13}+1)(y_{24}+1) - \lambda(y_{34}+\lambda/2)\} + \\ + (\lambda^2/4 - 1)(y_{34} - \lambda/2)^2 . \tag{2.1}$$

We call this curve Γ; following the method of Tarski [6], Γ can be plotted on the real y_{24}-y_{13} plane. The configuration depends on whether $y_{34} \gtrless -\lambda/2$. In the limit described above, with M fixed at its actual value and λ decreasing to 2, we always end up in the region $y_{34} < -\lambda/2$; the corresponding configur-

ation of Γ is shown (*) in Fig. 3 (a). Fig. 3 (b), consisting of two straight lines and a hyperbola, shows the limiting configuration as $\lambda \to 2$; it can be checked trivially from (2.1). It arises from Fig. 3(a) by the coalescence of L_1 and L_2 at the value y_{34}, while the right-hand vertical tangent (which is non-singular) moves down to $+1$.

Recalling that the limit $y_{34} - i\varepsilon$ is always understood, we are now ready to discuss the singular nature of the arcs of Γ as shown in Fig. 3 (a). We note

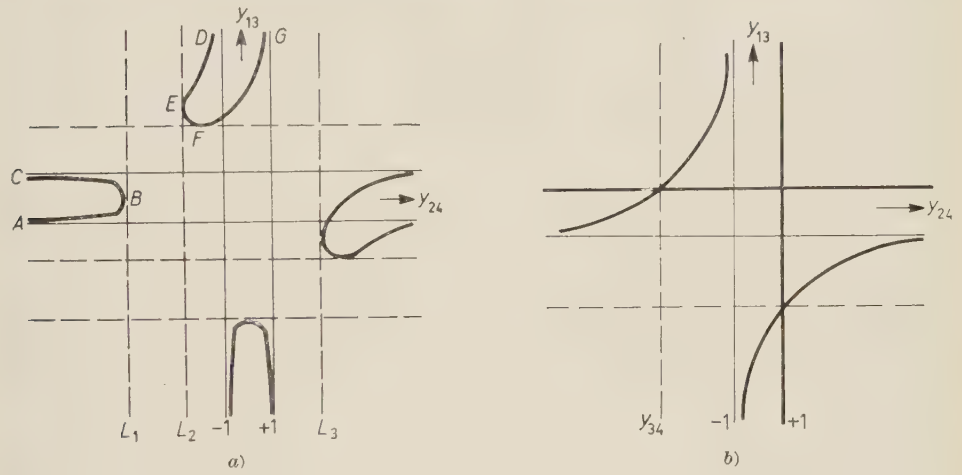


Fig. 3. – a) The curve Γ for $y_{34} < -\lambda/2$. b) Same as (a), in the limit $\lambda = 2$.

at the outset certain unconventional aspects: each arc lies entirely in the non-cut region as regards one of the variables; each is asymptotic at one end to a normal threshold and at the other to a non-singular line; and each is tangent at one and only one point to a line of vertex singularities, which is actually singular only in the non-physical limit (*i.e.* on the non-physical side of the normal branch cut).

Because of the symmetry in y_{24} and y_{13} it suffices to discuss the left-hand and the top arcs. Using standard theorems (**) ([6,8]) we conclude the following:

i) The arc AB is singular in the unphysical limit $y_{24} + i\varepsilon$; since it lies entirely in the uncut part of the y_{13} axis, no question arises about the sign of the imaginary part of y_{13}.

(*) Figs. 3a and 4 are intended to illustrate only the general configuration of Γ. In drawing them no attempt has been made to locate the points of inflection in such a way as to preserve the important property of fourth order curves, that they can be intersected by a straight line in at most four points. This property would be important for a detailed enumeration of the complex surfaces joining the real arcs Γ.

(**) Cf. for instance theorem 3.2 of ref. ([7]).

ii) The arc BC is entirely non-singular.

iii) The arc DE is singular in the unphysical limit $y_{24} - i\varepsilon$: as in i) above, no limiting prescription is needed for y_{13}.

iv) The arcs EF and FG are non-singular, (the line $y_{13} = L_3$ is non-singular so that contact with it at F has no effect on Γ).

Only now are we in a position to interpret the limiting configuration shown in Fig. 3 (b) where $\lambda - 2$, which alone is relevant to the analytic properties of the actual amplitude F. We note that each hyperbolic arc lies entirely in the cut portion of the plane as regards one variable, and in the uncut portion as regards the other variable; each arc is singular on the unphysical edge of the cut in the relevant variable. The straight lines are non-singular.

2·3. *Singularities in the region of complex y_{13} and y_{24}.* – The standard search-line method ([6]) can now be applied to determine the singular nature of the complex Landau surfaces (2.1) which are joined to the arcs of the real section Γ shown in Fig. 3 (b). Because of the necessary assignment of an infinitesimal imaginary part to y_{34} (*i.e.* because y_{34} lies on its normal branch cut), we must now distinguish four types of surfaces, on which the signs of the imaginary parts of y_{13} and y_{24} are, respectively, (++), (+−), (−+) and (−−).

The singular arcs of Γ all have positive slope, so that on the complex surfaces joined to them y_{13} and y_{24} have imaginary parts of like sign. Further, the arcs of Γ are singular only on the unphysical edge of the relevant cut; hence of the two complex surfaces leaving these arcs, the surface of type (−−) is not singular but the surface of type (++) is singular.

The existence of this surface of singularities for real y_{34} and complex y_{13} and y_{24} shows at once that a two-dimensional Mandelstam-type representation of the decay amplitude, with cuts only along the real axes, is incorrect. *A fortiori* one cannot write a three-dimensional representation in s, t, and M^2, with only real cuts, of the type proposed by BONNEVAY ([9]).

2·4. *Variation of M^2.* – It may be worth while to comment briefly on our results from the point of view of an attempted continuation of F in M^2 around its branch point dividing the stable from the unstable regions (*).

([9]) G. BONNEVAY: *Proc. of the 1960 Rochester Conference*, p. 523.

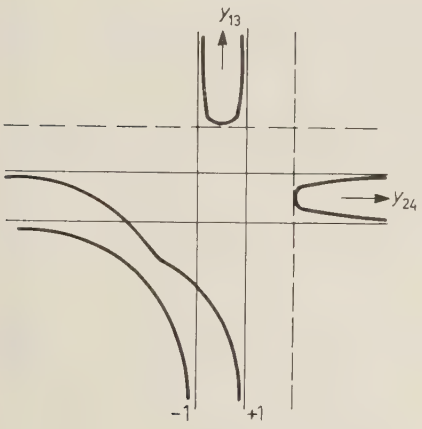
(*) Apart from the necessity of the limiting procedure in λ, this is part of the general problem of determining the analytic properties of a single loop four-point function with one externally unstable vertex. This has been considered by COOK and TARSKI ([7]) in an Appendix of their paper on the five-point function. Our conclusions agree with theirs wherever they overlap. In particular, they also discuss our Fig. 4c below.

If M is assigned a stable value, $M < 3$, then F does possess a Mandelstam-type representation. The boundary curve of the spectral function is the lower left-hand arc of the hyperbola

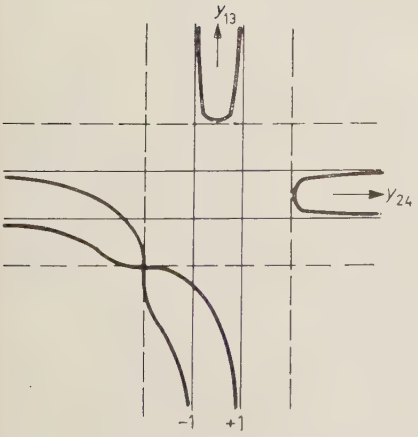
$$(y_{13} + 1)(y_{24} + 1) = 2(y_{34} + 1)\,,$$

obtained from (2.1) by setting $\lambda = 2$. As y_{34} decreases to -1 which now, with $\lambda = 2$, corresponds to the true instability point $M = 3$, the hyperbola degenerates into its asymptotes, and for $y_{34} < -1$, it opens up again in the opposite pair of quadrants as shown in Fig. 3 (*b*). We feel that, in the absence of theorems much more powerful then any that we are aware of, it is unlikely that such behaviour could be interpreted easily without making the fairly wide detour of lifting the degeneracy for $\lambda = 2$, and considering the vertex singularities in detail.

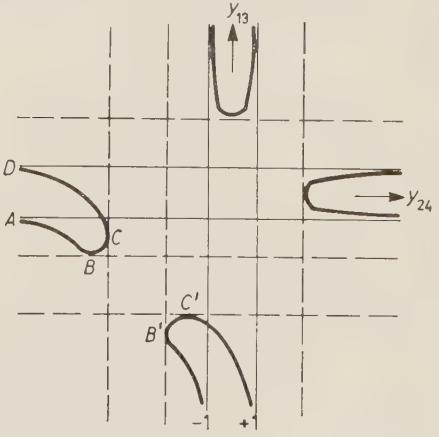
The fact that as M increases through the instability point, an entire complex surface between the two arcs suddenly becomes singular (as do the hitherto non-singular real portions of the hyperbola) may at first sight appear to be paradoxical, even though $M = 3$ is known to be a branch point of the amplitude. One can gain some insight into the detailed mechanism responsible for this behaviour if one realizes that it



a)



b)



c)

Fig. 4. – *a*) The curve Γ for $-1 > y_{34} > -1$; *b*) for $y_{34} = -1$; *c*) for $-1 > y_{34} > -\lambda/2$.

is reasonable to expect the transition between the two cases to be gradual only if it is envisaged as happening for fixed λ, whereas in order to reproduce the analytic properties of the physical amplitude the limit $\lambda \to 2+$ must be taken separately for each value of M as it is increased past the instability point.

Indeed, when M increases for fixed $\lambda > 2$, we have, for M just below $(\lambda+1)$, *i.e.* $1 > y_{34} > -1$, the configuration shown in Fig. 4 (*a*). At the formal instability point for fixed λ, *i.e.* at $M = (\lambda+1)$, $y_{34} = -1$, we have the configuration shown in Fig. 4 (*b*). When M is increased just past $(\lambda-1)$, we are in the region $-\lambda/2 < y_{34} < -1$, corresponding to the configuration (7) of Fig. 4 (*c*). Here, the line L_1 is singular on the unphysical edge of the cut, as it was in Fig. 3 (*a*), but L_2 is now singular on the physical edge (1). It follows that in the $(--)$ limit the arc AB is singular and BCD is not, while in the $(++)$ limit ABC is singular and CD is not. When M passes the formal instability point $M = (\lambda+1)$, the surface of complex singularities does, as expected, begin to spread out gradually from the double point featured in Fig. 4 (*b*): it links the arcs BC and $B'C'$ of Fig. 4 (*c*). The transition from the configuration of Fig. 4 (*c*) to that of Fig. 3 (*a*) can be traced by following the motion of the line L_2, which, as M increases (or as λ decreases), first moves into coincidence with the line -1 at $y_{34} < -\lambda/2$ (where its singularity changes from the physical to the unphysical edge of the normal branch cut), and then moves back in the negative direction.

Thus it is the final limiting procedure $\lambda \to 2+$ which, with M now fixed at any value above 3, always transfers us into the region $y_{34} < -\lambda/2$ whose configuration is that of Fig. 3 (*a*); it is this limiting procedure which brings about the abrupt change in the actual amplitude F as M passes the true instability point at $M = 3$. Note that the configuration of Fig. 4 (*c*) is obviously incapable of merging continuously into the known degenerate configuration of Fig. 3 (*b*) without first passing through that of Fig. 3 (*a*).

2·5. *One-dimensional representation.* – Even though the decay amplitude does not possess a Mandelstam representation, it is perhaps of some limited interest to know whether the partial amplitudes we have been considering do at least satisfy a one-dimensional representation with only real cuts. In considering this question we fix y_{13}, say, at some real value (not necessarily in the physical decay region) and ask for the analytic properties of f as a function of y_{24}. For $\lambda > 2$ it is evident from Fig. 3 (*a*) that with y_{13} real it is only for $L_3 > y_{13} > 1$ and for $L_2 > y_{13} > L_1$ that f could have singularities for complex y_{24}. As $\lambda \to 2$ these dangerous regions shrink to zero in Fig. 3 (*b*). Hence for real fixed y_{13} F has no singularities for complex y_{14}, and a one-dimensional dispersion representation with real cuts only is indeed valid.

If y_{13} is to be fixed at a value below -1 (*i.e.* on its cut), then in order for

the dispersion integral to represent the physical amplitude, y_{13} must be placed on the physical side of its cut. But in that limit the right-hand (and lower) arc of the hyperbola in Fig. 3 *b*) is actually non-singular; thus as far as the one-dimensional representation is concerned it can be left completely out of account. We conclude that for this diagram there are no scattering singularities in y_{24} for fixed $y_{13} < -1$; for fixed $y_{13} = \tilde{y}_{13} > -1$, our particular diagram has a singularity at the value of y_{24} given by the intersection of the line $y_{13} = \tilde{y}_{13}$ with the upper (left-hand) arc, say at $y_{24} = \tilde{y}_{24} < -1$. We know that this singularity lies on the unphysical side of the y_{24} cut. The one-dimensional representation of our diagram has exactly the same form as that of the vertex and self-energy diagrams, though this simplicity is in a sense deceptive, because there is no easy or physically illuminating way to continue its spectral function in the fixed (momentum-transfer type) variable. The physical relevance of these conclusions is better seen when they are expressed in terms of the *s*-variables, which is done in the next subsection.

2·6. *The one-dimensional representation in terms of the s-variables.* – The *y*-variables describe individual diagrams in the terms best suited to their own particular analytic properties. There are actually six different contributing diagrams, which can all be described by the same equations in terms of the *y*-variables, but with different relations between the *y*- and the *s*-variables. We now re-express our results in terms of the *s* and locate the singularities found in the last subsection relative to the physical decay region. We confine ourselves to the two diagrams whose natural variables are *s* and *t*; the information from the other diagrams can be obtained straightforwardly by first formulating it in terms of the natural variables for each diagram and then using the constraint (1.1). Note that by inquiring into a one-dimensional representation in one variable with one other variable fixed we are, of course, destroying the formal symmetry of the amplitude in *s*, *t*, and *u*.

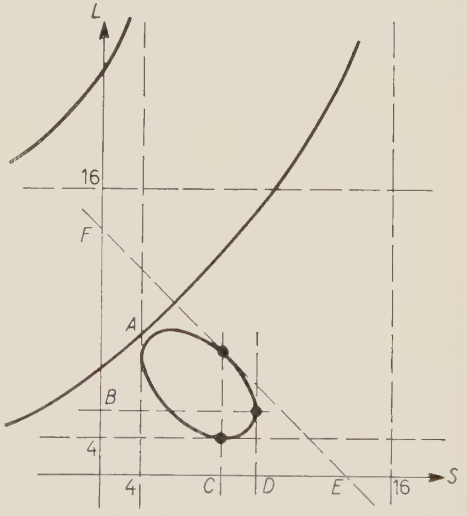


Fig. 5. – The physical decay region and the curves of singularities relevant to the dispersion representation in *t* for fixed *s*. The line EF is $s + t = M^2 - 1$, corresponding to $u = 4$. The points A, B lie at $t = 4 + 2(M^2 - 9)/3$ and at $(M+1)$. The points C, D lie at $s = (M^2 - 1)/2$ and at $(M-1)^2$.

The physical decay region is the interior of the oval shown in Fig. 5. The singular arc of Fig. 3 (*b*) is the

upper arc of Fig. 5, asymptotic to the normal thresholds appropriate to the diagram of Fig. 1 (*a*). Its equation is

$$(s-4)(t-16) = -8(M^2-9)\,.$$

The other diagram whose natural variables are s and t is the one obtained by interchanging s and t in Fig. 1 *a*); its singular curve yields the lower of the two arcs (*) in Fig. 5, whose equation is

$$(s-16)(t-4) = -8(M^2-9)\,.$$

For fixed $s=\tilde{s}$, we get the one dimensional representation in t:

$$F(\tilde{s}+i\eta,\, M^2+i\eta,\, t) = \int_4^\infty \frac{\varrho(\tilde{s}+i\eta,\, M^2+i\eta,\, t')\,\mathrm{d}t'}{t'-t-i\varepsilon}\,.$$

If the line $s=\tilde{s}$ intersects one or both of the singular arcs, say at $t=\tilde{t}$, then there is a singularity of F at $\tilde{t}-i\varepsilon$ on the unphysical edge of the cut. This corresponds to a singularity of the spectral function on the upper (physical) edge of the cut; a similar case has been discussed in I. Note that if s is fixed in then physical decay region, then only the lower arc is relevant, and the singularity on the *lower* edge of the cut lies at or above the point $t=4+2(M^2-9)/3$ *i.e.* a finite distance to the right of the threshold and never in the immediate vicinity of the physical region which extends from $t=4$ to $t=(M-1)^2$ along the *upper* edge of the cut.

2·7. *Nature of the singularities.* – The influence of any singularity on the amplitude depends both on its nature and on how far it is from the physical region. We now enquire what is the type of the singularities that we have been discussing. A scattering singularity is normally of type $z^{-\frac{1}{2}}$, z being any variable that vanishes as the singularity is approached. In our case this is the nature of the singularity of f with fixed λ_1^2 and λ_2^2. The two integrations over λ_1^2 and λ_2^2 each increase the index by unity, so that the singularity of F is of type $z^{\frac{3}{2}}$. This is what one expects from the fact that all vertices of our diagrams have three internal lines. Since these scattering terms enter addi-

(*) The exact relative location, in the plane, of the singular arcs and of the boundary curve of the physical region are not crucial, since they lie on opposite sides of the cut, as explained below in the text. Fig. 5 is not necessarily correct in showing them as nonintersecting. In the non-relativistic limit (working to second order in $(M-3)$) the boundary curve becomes an ellipse and the lower hyperbolic arc does intersect it.

tively into an amplitude which, as we shall discuss in the next section, is most probably dominated by quite different terms, and since the singularity is of a type at which the singular components vanish, their quantitative effects remain likely to be negligible.

3. – Conclusions.

The total kinetic energy available to the pions in $K \to 3\pi$ decay being relatively low, it is not implausible on physical grounds that only S-wave interactions are important, in which case the single-s-dependent terms would dominate the amplitude. These are of two kinds, some independent of and some strongly dependent on M. We found in I that the leading terms of both kinds have one-dimensional dispersion representations with normal thresholds; there appeared no mathematical reason to suppose that one kind is much more important than the other apart from an appeal to the smallness of the pion-pion coupling constant.

We are aware of only one other physical argument against the importance of the M-dependent terms. These arise exclusively from diagrams in which each of the pions emerging from the elementary $K\pi^3$ vertex interacts at least once more ([1,4]). While there is no kinematic reason why two of the pions leaving the vertex should not travel together in the same direction and therefore scatter each other strongly, with the third pion leaving in the opposite direction without interacting, it could be argued that there is no possible arrangement of this kind whereby all three of the pions can spend a long time in each other's vicinity. In our opinion the plausibility of this argument is decreased by the observation that actually the pion-pion interaction has finite range (half a pion Compton wavelength), and that the kinetic energy of the pions being low, they do not leave each other's range of influence instantaneously. This may be the reason why the comparison in I of the leading terms of the two kinds of vertex contribution failed to indicate any dominance of one kind over the other. It is important to realize that this question does not, of course, reflect on the validity of any treatment of the decay confined to such single-s-dependent terms, but which does not rely on a continuation in M^2 ([10]).

In contrast to the single-s-dependent terms, which may or may not depend on M, all terms that are functions of two of the s are crucially M-dependent. We have seen that their behaviour cannot readily be inferred from the case of stable M, and that in general they are quite intractable from any practical viewpoint. On the other hand, it appears very likely that for interpreting the present data from a quantitative point of view these terms are negligible.

([10]) N. N. Khuri and S. B. Treiman: *Phys. Rev.*, **119**, 1115 (1960).

In our own view our results are most profitably looked at as a cautionary example of the pitfalls which may beset one in making conjectures about the analytic properties of probability amplitudes in field theory.

* * *

We welcome this opportunity to acknowledge discussions with Dr. L. F. COOK, Dr. J. TARSKI, and Dr. A. C. T. WU. The first named author is grateful also to the Governing Body of Christ Church for their continued leave of absence, and to the Physics Department at Brookhaven for its hospitality.

Notes added in proof.

1. – G. BONNEVAY (*Some analytic properties of three-pion decay amplitudes*, Birmingham, preprint) has independently come to conclusions identical with ours. He notes in addition that if at least one of the variables s and t is confined to its upper half plane, then subject to this explicit restriction the amplitude has a Mandelstam representation. Thus there exists a sheet on which the function has no complex singularities; but this sheet overlaps the physical sheet only in part.

In the same paper, BONNEVAY gives an analytic proof that the exact boundary and Landau curves can at most touch but never intersect.

We are grateful to Dr. BONNEVAY for sending us a preprint of his work.

2. – Fig. 4(*b*) is incorrectly drawn. Actually at the double point both tangents have finite slopes.

RIASSUNTO (*)

Studiamo le proprietà analitiche dei più semplici diagrammi di Feynman ad una sola ansa il cui contributo al decadimento $K \to 3\pi$ dipende da entrambe le variabili indipendenti dell'energia. Questo studio viene effettuato sostituendo le due linee pioniche doppie interne con linee singole fittizie aventi masse di riposo discrete λ_1 e λ_2 ed integrando in λ_1 e λ_2 alla fine del calcolo. Le proprietà analitiche dell'ampiezza effettiva sono quelle dell'ampiezza fittizia al limite in cui λ_1 e λ_2 tendono dal di sopra indipendentemente al valore di due masse pioniche. L'ampiezza ha delle singolarità complesse e non soddisfa ad una rappresentazione del tipo di Mandelstam. Tuttavia una rappresentazione unidimensionale della dispersione solo con tagli reali vale per una variabile se l'altra variabile viene fissata ad un valore reale qualunque. Facciamo brevi commenti sul cambiamento delle proprietà dell'ampiezza quando la massa del mesone K cresce oltre il punto di instabilità.

(*) *Traduzione a cura della Redazione.*

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Analytic Properties of the Three Point Function.

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(ricevuto il 22 Giugno 1961)

Summary. — For the vertex function in lowest order, the part of the Landau surface which is singular on the physical sheet is identified for the general case of three complex external masses. The result is extended for the seven other sheets obtained by crossing the normal (two-particle) branch cuts. Applications to production processes and reactions involving resonant or unstable states are indicated.

Introduction.

In this paper, we shall give a complete description of the location of the singularities of the vertex function in lowest order perturbation theory, considered as a function of three complex variables. Although the main results are probably known to specialists in this field (1), we think that the treatment provides an instructive illustration of the continuity arguments of Tarski (1a), Polkinghorne and Screaton (2) and Eden (3) and others; also, the results are needed in the study of production amplitudes or decay processes.

After a short description of the general method (Section **1**) which is applicable to diagrams of any order, in Section **2** we study the properties of the vertex function on the physical sheet. In Section **3** we extend the result for unphysical sheets and in Section **4** we discuss possible applications.

(1) G. Källén and A. Wightman: *Mat. Fys. Skr. Dan. Vid. Selsk.*, **1**, 6 (1958).
(1a) J. Tarski: *Journ. Math. Phys.*, **1**, 154 (1960).
(2) J. C. Polkinghorne and G. R. Screaton: *Nuovo Cimento*, **15**, 289, 925 (1960).
(3) R. J. Eden: *Phys. Rev.*, **121**, 1567 (1961).

1. – General methods.

The possible singularities of a given Feynman graph are those associated with all possible contractions of it (« lower order » singularities), together with the « leading » singularity, which comes from the (uncontracted) diagram itself. Assuming we know the positions of all the lower order singularities, we define the « physical sheet », in the complex space of the external invariants, by cuts starting from the singularities. The Landau conditions [4] for the uncontracted diagram then define a certain surface Σ in this space. The problem is to pick out the part of Σ which is singular inside the physical sheet; to save repetition in the following this will be called simply the singular part, and the remainder the non-singular part.

The procedure is to find first one point P of Σ which is non-singular; then, by continuity, all points of Σ which can be joined to P by lines on Σ which do not cross a cut are also non-singular. We see, then, that the only parts of Σ which can possibly be singular are those which can only be reached from P by crossing at least one cut. To know if such parts are indeed singular, it is sufficient to examine one point inside each part. That is, the cuts make a separation of Σ into singular and non-singular regions. Of course, there is initially a certain arbitrariness in the choice of the cuts, but they should be chosen such that they give the right Feynman conditions when the external invariants are in the physical region for any process represented by the given graph.

2. – The vertex function.

We discuss the singularities of the Feynman amplitude F, associated with the graph Fig. 1 *a*), considering it as a function of the three external invariants p_i^2, real or complex. That is, we study F in a 6-dimensional space. Using the familiar notation [5], we define $p_1^2 = m_2^2 + m_3^2 - 2m_2m_3x$, and p_2^2, p_3^2 cyclically in terms of y and z. Apart from inessential factors, F is then given by

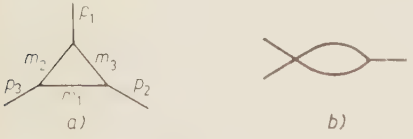

Fig. 1.

$$(1) \qquad F = \int_0^1\int_0^1\int_0^1 \frac{\delta(1 - \alpha_1 \cdots \alpha_2 - \alpha_3)\,\mathrm{d}\alpha_1\mathrm{d}\alpha_2\mathrm{d}\alpha_3}{[\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + 2\alpha_2\alpha_3 x + 2\alpha_3\alpha_1 y + 2\alpha_1\alpha_2 z]^2}\,.$$

[4] L. D. LANDAU: *Nucl. Phys.*, **13**, 181 (1959).

[5] R. KARPLUS, C. M. SOMMERFIELD and E. H. WICHMANN: *Phys. Rev.*, **111**, 1187 (1958).

The condition for a leading singularity is

$$(2) \qquad \begin{vmatrix} 1 & x & y \\ x & 1 & z \\ y & z & 1 \end{vmatrix} = 0 ,$$

and this is the equation of Σ, namely

$$x^2 + y^2 + z^2 - 2xyz - 1 = 0 .$$

The lower order singularities are given by the vanishing of the leading minors:

$$x = \pm 1 , \qquad y = \pm 1 , \qquad z = \pm 1 .$$

By examining the denominator D in (1), it is clear that none of x, y, $z = +1$ is a singularity, while all of x, y, $z = -1$ are. The latter correspond to contracted diagrams of the type Fig. 1 *b*). According to our general prescriptions in the [6]-space of (xyz) we have to define three cuts starting from these three branch points.

Let us write $X_1 = \mathrm{Re}\, x$, $X_2 = \mathrm{Im}\, x$ and similarly for y, z.

We define the cuts by

$$(3) \qquad \{X_1 \leqslant -1,\ X_2 = 0\}, \qquad \{Y_1 \leqslant -1,\ Y_2 = 0\}, \qquad \{Z_1 \leqslant -1,\ Z_2 = 0\}.$$

We know that when two of the variables correspond to real physical masses, the cut in the third variable lies on the negative real axis. When these two variables become complex the position of the cut in the third variable is a matter of definition.

The surface Σ in the [6]-space is a [4]-variety defined by

$$(4) \qquad \begin{cases} X_1^2 + Y_1^2 + Z_1^2 - 2X_1Y_1Z_1 - 1 - \\ \qquad\qquad - (X_2^2 + Y_2^2 + Z_2^2 - 2X_1Y_2Z_2 - 2X_2Y_1Z_2 - 2X_2Y_2Z_1) = 0 , \\ X_1X_2 + Y_1Y_2 + Z_1Z_2 - X_2Y_1Z_1 - X_1Y_2Z_1 - X_1Y_1Z_2 = 0 . \end{cases}$$

Each of the cuts (3) is half an Euclidean [5]-space which intersects Σ on a [3]-variety which separates the singular regions of Σ from the non singular ones.

2·1. *One variable fixed and real.* – First let us fix one external variable z at a real value, say $Z_1 = a$, $Z_2 = 0$, with $a > -1$ so as to avoid the cut in z.

Σ is now a [2]-variety embedded in the [4]-space $(X_1 X_2 Y_1 Y_2)$. We can give a brief geometrical descritpion of it.

Take a in the range $-1 < a < 1$. The intersection of Σ by the subspace $Y_2 = 0$ is, by (4), a curve of the fourth degree degenerated into an ellipse in the (X_1, Y_1) plane and a hyperbola in the plane $(aY_1 - X_1 = 0,\ Y_2 = 0)$, which is perpendicular to the (X_1, Y_1) plane and passes through the tangent points $x = \pm a$. The intersection of Σ and $X_2 = 0$ is similar, containing the same ellipse; so is the intersection of Σ and any [3]-sub-space $\alpha X_2 - \beta Y_2 = 0$ rotating around the intersection of $X_2 = 0$ and $Y_2 = 0$, that is, around the plane (X_1, Y_1). Fig. 2 represents the projection of Σ on the [3]-subspace $Y_2 = 0$. In this projection the hyperbola $[X_2 = 0]$ is degenerated into the double line l.

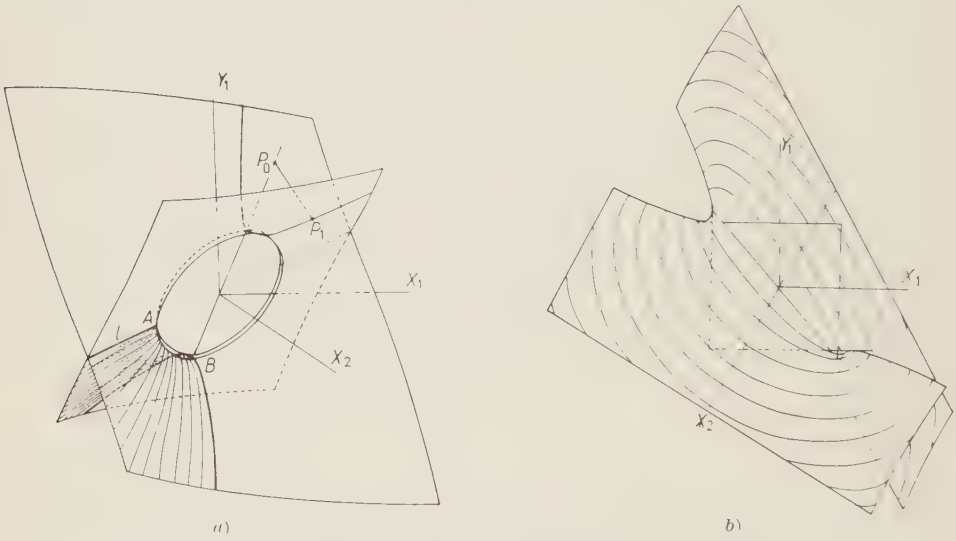

Fig. 2. – Projections in 3 dimensions of Σ showing the shaded region. *a*) Z real beween -1 and 1 (l is only a double line due to the projection). *b*) Z real less than -1. Σ should be completed by symmetry in the $X_1 Y_1$ plane. Only the shaded region (half of Σ) is represented. The ellipses are intersections of Σ by the subspaces $\alpha X_2 + \beta Y_2 = 0$.

Now, the cuts we have defined above divide Σ into two regions separated by the thick lines in Fig. 2*a*). To find which part of Σ is possibly singular, we start at a point $P^{(0)}: (x^{(0)}, y^{(0)})$ not on Σ, such that $X_1^{(0)},\ Y_1^{(0)} > 1$, $X_2^{(0)},\ Y_2^{(0)} = 0$. At $P^{(0)}$, F is non-singular. We now increase $X_2^{(0)}$ till the point P on Σ is reached. Along this path, no deformation of the α-contours is necessary since D never vanishes in the domain of integration. So F is well-defined by (1) at P, and is non-singular there. It follows by continuity that the only part of Σ which can be singular is the shaded region.

The only way to enter this region without crossing a cut or leaving Σ is to pass through one of the singular points on Σ, $x=-1$ or $y=-1$ (A and B on Fig. 2*a*)). Let us consider the point (xyz) on Σ. Then, in the α [4]-space (there are two independent α's, both complex) the surface $D=0$ degenerates into two planes meeting in a point $M(xyz)$ (coincident singularity). The point (xyz) is non-singular as long as M is not inside the region of integration. When we cross the point $x=-1$ or $y=-1$, M penetrates the integration domain (6); along the arc AB of Fig. 2; M is inside the integration domain and the points of the arc are singular, so that, by continuity, all the shaded region is singular also, despite the fact that M is then no longer inside the original integration domain.

Next, consider the case $a>1$. The surface can be traced out as before. In this case all the intersections of Σ by the [3]-subspaces $\alpha X_2+\beta Y_2=0$ contain the same real hyperbola in the (X_1, Y_1) plane, the branch $L:[X_1 \leqslant -1, Y_1 \leqslant -1]$ of which is therefore the only line of separation on Σ. As before, the point P is non-singular and so are all points which can be reached from P without crossing L or leaving Σ. But now any point on Σ can be reached from P, including any point of L itself, so that Σ is entirely non-singular. In a part of L, the double point M comes into the domain of integration; but we can reach such a point, following Σ, without passing through the border of this domain, so that although the coincident singularity comes inside the integration region, it does not pinch the contours and the corresponding point (xyz) on Σ is not singular.

To consider the case $a<-1$, it is necessary to give a small imaginary part to a, since F has a cut from $a=-1$ to $a=-\infty$. At this stage we must therefore consider F as a function of three complex variables.

2·2. *All variables complex.* – We know already from Section 2·1 the singular character of some points on Σ, and we shall use this to find a starting point P on Σ which is non-singular. Clearly it is now not so easy to represent Σ on paper. For our purposes, it will be enough to consider the projections of it onto the x, y and z planes separately. For some given z, Σ may be considered as a mapping of the x and y planes onto each other.

We must be careful to remember, however, that the mapping of, for example, the y-plane onto the x-plane is a double one:

$$x = yz \pm [(1-y^2)(1-z^2)]^{\frac{1}{2}}, \tag{5}$$

in other words a point y in the y-plane is actually the projection of *two* points on Σ which have two different x-projections.

(6) M. Fowler, P. V. Landshoff and R. W. Lardner: *Nuovo Cimento*, **17**, 956 (1960).

One can show from eq. (4) that the image in Σ, on the x-plane, of the real y axis is a degenerate quartic, made up of a hyperbola and an ellipse, foci $x = \mp 1$, and passing through the points $x = \mp z$ (Fig. 3). The same is true,

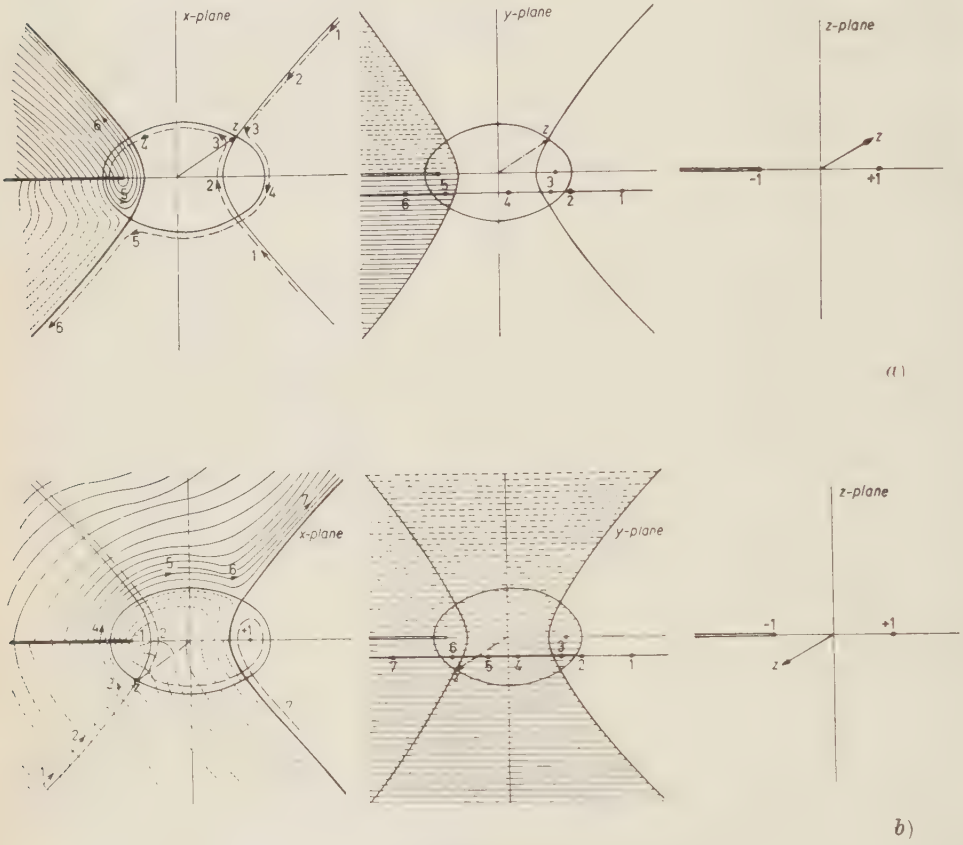

Fig. 3.

of course, of x and y interchanged; in the following though, for definiteness, we shall continue to consider only the mapping of the y-plane onto the x-plane. For z real and greater than 1, or less than -1, the hyperbola collapses onto the real axis, while for z real, $-1 < z < 1$, the ellipse collapses. When $Z_1 > 0$ the line of separation on Σ coming from the cut $[Y_1 = -1,\ Y_2 = 0]$ is the left branch of the hyperbola (Fig. 3 *a*)); when $Z_1 = 0$ the two branches collapse onto the imaginary axis, and when $Z_1 < 0$ the right branch is the line of separation between singular and non-singular parts of Σ (Fig. 3 *b*)). Of course, the line of separation $[X_1 = -1,\ X_2 = 0]$ is also a branch of a hyperbola which, in projection on the plane, collapses onto the real negative axis $X_1 \leqslant -1$.

To have a complete description of Σ, instead of doing cumbersome projections onto other planes, it is enough to follow the mapping Σ along simple lines like those which project on the y plane into straight lines parallel to the real y axis. Fig. 3 *a*) represents the image in Σ of a line near the real y axis, with numbers to indicate qualitatively the correspondence as Y_1 decreases from $+\infty$ to $-\infty$.

This method enables us to see how the two dividing lines effectively separate Σ into two parts, which are represented in the x and y projections in Fig. 3 *a*) and 3 *b*), for $Z_1 > 0$, $Z_1 < 0$ respectively. Moreover, when y penetrates the shaded (either hatched or dotted) region, only *one* of the two corresponding x's is in the shaded region. We shall call a point « in the shaded region » if *both* its projections on the x and y planes are in the shaded region.

To find which part of Σ is singular, we again have first to find a point P which is non singular. This we can do using the results of Section **1**. For example, we know that for y and z real and greater than 1, say $y^{(0)}$, $z^{(0)}$, there are two points on Σ, say $P^{(1)}:(x^{(1)}, y^{(0)}, z^{(0)})$ and $P^{(2)}:(x^{(2)}, y^{(0)}, z^{(0)})$ neither of which is singular and for both of which x is real and greater than 1. We now increase $Z_2^{(0)}$, $y^{(0)}$ being kept fixed, so that we get a figure like Fig. 3 *a*), with a dividing hyperbola branch to the left. Then, since during this motion $X_1^{(1)}$, $X_1^{(2)} > 0$, the points $P^{(1)}$, $P^{(2)}$ remain inside the unshaded region without ever crossing a line of separation.

Then, by continuity, the only part possibly singular is the shaded region. To find the character of this shaded region it is enough to know the character of one of its points: let us take $z = a$, a real and $-1 < a < 1$. From the result of Section 2·1 we know that the points of the arc AB of Fig. 2 are in the shaded region and singular. But, for any z, we can join any point of the corresponding shaded region to a point of AB by a continuous path traced on Σ, and avoiding any cut.

Hence we have seen that for any complex value of z, the corresponding shaded region is the only singular part of Σ; and this gives us the complete location of the singularities of $F(xyz)$.

We may summarize the results of this section as follows. Suppose we have a problem in which two of the variables, say y and z, are fixed and we are interested in the analytical properties of F in x. For the given y and z (5) gives the two x's ($x^{\pm}$) which lie on Σ. For the given z we construct the dividing hyperbolae in the x and y planes, obtaining a figure of the type 3 *a*) or *b*), depending on the sign of Z_1. Then:

a) if y is not in the shaded region there is no anomalous singularity in x (neither of $x^{\pm}$ is singular);

b) if y is inside the shaded region, one of the $x^{\pm}$ is singular.

There are two other points to notice. First, the problem is of course com-

pletely symmetric in x, y and z. Second, when we have located an anomalous branch point x^+, say, the physical sheet, according to our general definition in Section **1**, must be defined by a cut starting from x^+.

3. – Analyticity properties on unphysical sheets.

The discussion of Section **2** relates to the physical sheet of $F(xyz)$—that is the topological product of three cut planes. Now, since it is known (7) that each of the normal thresholds is a square root branch point, that is it generates two sheets only, we can introduce seven other sheets for F. We call these eight sheets (PPP), (UPP), etc., and the corresponding functions F_{PPP}, F_{UPP}, etc., where the indices refer to the x, y and z planes and mean « physical » and « unphysical ».

The analytical properties of F on these unphysical sheets are of interest for at least two reasons. Firstly, knowledge of a scattering amplitude on an unphysical sheet gives us the properties of the absorptive part of the amplitude, defined in terms of the difference between the amplitude on the corresponding physical and unphysical sheets (8).

In our case, in the « x »-channel for example, the absorptive part is defined by

$$F_x = \frac{1}{2i}(F_{PPP} - F_{UPP}) .$$

Secondly, using perturbation theory in the extended sense of LANDAU (4) we can consider cases in which one of the particles in Fig. 1 is unstable (resonant state). The corresponding x-variable must then be taken complex, and on the unphysical sheet (*).

As an example, we shall discuss F_{UPP}. To know the singular character of any point A on Σ, we join it to a point B on Σ by a path on Σ which crosses only the x-cut once. From Section **2** we know the singular character of B on (PPP)—for example, suppose it is singular: that is, we find a singularity of F at B if, starting from an arbitrary point M on (PPP), we approach B by following a path which crosses no cut. By continuity A is singular also if we approach it from M by a path such that the curvilinear trian-

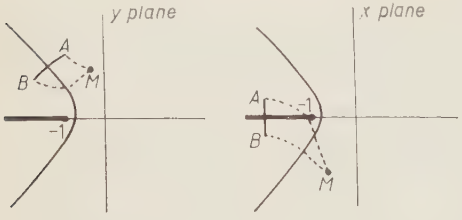

Fig. 4.

(7) R. OEHME: *Phys. Rev.*, **121**, 1840 (1961).
(8) R. E. CUTKOSKY: *Journ. Math. Phys.*, **1**, 429 (1960).
(*) One of us (G.B.) thanks Prof. STANGHELLINI for pointing this out to him.

gle ABM does not contain a branch point. The situation is illustrated in Fig. 4 and from this figure we see that MA crosses the x-cut without crossing any other cut. That is, penetrating (UPP) we find A is singular, while it was non-singular for (PPP).

By a repeated application of the same argument we find

a) the four functions F_{PPP}, F_{UUP}, F_{PUU}, F_{UPU} are singular in the shaded region and non-singular in the unshaded region;

b) the converse holds for the four functions F_{UUU}, F_{PPU}, F_{UPP}, F_{PUP}.

We remark that the « non-Landau » singularity discovered by CUTKOSKY [8] which occurs when $p_i^2 = (\sqrt{p_j^2} + \sqrt{p_k^2})^2$, never appears on the eight sheets we have discussed—that is to say provided we never encircle an *anomalous* branch point. But we must not forget that on the sheet UPP, the branch surface $x = +1$, becomes singular now, since the x cut has been crossed. In the same way, the surfaces $x = +1$, $y = +1$ are singular for F_{UUP}, etc.

4. – Applications.

4·1. *Production processes.* – The general description of Section **2** enables us to treat cases in which external instability holds at any vertex, and cases in which there is some relation between x, y and z. Both cases occur in production amplitudes.

Consider the amplitude for the five particle process $p_1 + p_2 \to p_3 + p_4 + p_5$ (the p_i's are four-momenta). Of the ten possible external invariants $p_i \cdot p_j$ only five are linearly independent: the remainder, labelled collectively S, are redundant. Suppose we want the analytic properties of the amplitude in a momentum transfer variable, say $\Delta^2 = (p_2 - p_5)^2$ the other variables being in the physical region. Among the contractions of higher order graphs, there will be vertex diagrams of the type of Fig. 1, with external variables I, Δ^2 and M^2, where M is the mass of one of the (stable) five particles. I may be an energy variable independent of Δ^2, or it may depend on Δ^2 and possibly M^2. Let us call x, y, z the variables associated with Δ^2, I, M^2. Then z is always real and between -1 and 1, and the situation for x and y is described by Fig. 2, or by Fig. 3 with the ellipse collapsed onto the real axis.

If I is an energy variable independent of Δ^2, y is real and $\leqslant -1$. We must, therefore, specify how we approach y, since it lies on the cut: this is determined by the Feynman rules for obtaining the physical function, which imply that y must be approached from the lower half plane, so that y is in the dotted region. We see that the root for x which is in the dotted region

(positive imaginary part) is singular. This result has also been given by LANDSHOFF and TREIMAN (9).

If, on the other hand, there is a relation between x, y and z because I is redundant, we have only to solve (2) and apply the rules of Section 2·2. This is true even if the relation is not linear. For example, to discuss the singularities of the total cross-section for the process it might be convenient to choose a scattering angle as an independent variable. Let us define $t=(p_1-p_4)^2$, $s=(p_3+p_4)^2$. If we take θ to be the scattering angle between particles 1 and 4 in the c.m.s. of particles 3 and 4 we find:

$$(6)\qquad t=m_1^2+m_4^2-\frac{1}{2s}\{(s-m_3^2+m_4^2)(s-m_1^2+\Delta^2)-$$

$$-([s-(m_3+m_4)^2][s-(m_3-m_4)^2][s-(m_1+\Delta)^2][s-(m_1-\Delta)^2])^{\frac{1}{2}}\cos\theta\},$$

so that one of s and t is redundant, and the relation is non-linear.

4·2. *Reactions involving unstable particles (resonant states).* – In the isobar model of pion production in π-$\mathcal{N}$ scattering one has to consider anomalous thresholds coming from the graphs of Fig. 5 *a*), *b*). For Fig. 5 *a*) one needs the properties of F_{UPP} and for Fig. 5 *b*) those of F_{UUP}; in the latter case the singularities are the same as those of F_{PPP} and one finds that there is an anomalous threshold inside the physical region.

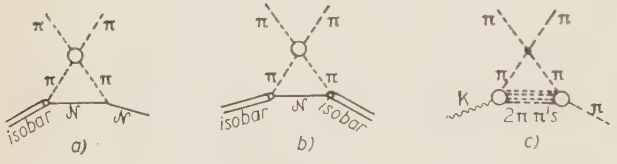

Fig. 5.

In the $K\to 3\pi$ decay, for the graph shown in Fig. 5 *c*) internal instability holds and there is no anomalous singularity. Then, as was pointed out by G. BARTON and C. KACSER (10), one can write a double dispersion relation of the Mandelstam type for the square of the K-mass and for the square of the energy $s=(p_1+p_2)^2$.

Despite the fact that the K is unstable, to first order in the weak interaction its lifetime must be considered as infinite compared with the time of interaction between the final pions, and it is correct to neglect the imaginary part of its mass (*). However, for a strongly interacting resonant state, like a supposed three pion resonance, the mass variable should be considered as

(9) P. V. LANDSHOFF and S. B. TREIMAN: *Nuovo Cimento*, **19**, 1249 (1961).
(10) G. BARTON and C. KACSER: preprint (Princeton, 1961).
(*) This argument has been given to one of us by Professor Stroffolini.

in the unphysical sheet, and anomalous thresholds would appear for the energy s.

* * *

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RIASSUNTO (*)

Nelle funzioni di vertice dell'ordine inferiore, identifichiamo, per il caso generale di tre masse esterne complesse, quella parte della superficie di Landau, che è singolare sul foglietto fisico. Estendiamo il risultato agli altri sette foglietti ottenuti attraversando i tagli normali dei rami (a due particelle). Indichiamo le applicazioni ai processi di produzione ed alle reazioni che coinvolgono stati risonanti od instabili.

(*) *Traduzione a cura della Redazione.*

Some Properties of High Angular Momentum Partial Wave Scattering Amplitudes.

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(ricevuto il 22 Giugno 1961)

Summary. — The form of the amplitude at low energy is shown to determine $l-1$ moments of the function, in the partial wave dispersion relation, which is the discontinuity across the unphysical branch cut. An inequality is given for the number of zeros in this function. The form of the moment sequence necessary to give a partial wave scattering amplitude consistent with unitarity and the partial wave dispersion relations is investigated. It is suggested that the sequence must be such that the absorption is a slowly varying function of the impact parameter.

1. – Introduction.

The purpose of this note is to examine some properties of high angular momentum partial wave amplitudes for nucleon-nucleon scattering. The starting point is the partial wave dispersion relation ([1,2]) in the variable $\omega = k^2$, k being the momentum in the centre of mass system. The nucleons are assumed to have spin 0 and to be of equal mass M. The partial wave amplitude is an analytic function in the ω plane cut from $\omega = 0$ to $+\infty$ and from $-\infty$ to $-\mu^2/4$, μ being the pion mass. The discontinuity across the right-hand branch cut is proportional to the imaginary part of the physical amplitude, but the discontinuity across the left-hand (unphysical) branch cut is unknown. The form of the partial wave amplitude near $\omega = 0$ (it is proportional to ω^l, where l is the angular momentum) is shown to determine the first $l-1$ moments of the discontinuity across the unphysical branch cut.

(*) Central Electricity Generating Board Research Fellow, St. Catherine's College.

(1) G. F. Chew: *Dispersion Relations* (ed. by G. R. Screaton) (London, 1961),

(2) S. W. Macdowell: *Phys. Rev.*, **116**, 774 (1960).

In Section 2, these moments are shown to be part of a Hausdorff moment sequence so that an inequality can be stated for the number of zeros in the discontinuity across the unphysical branch cut. The problem of constructing the real part of the amplitude from the moments is considered in Section 3. A further problem is the range of ω for which the real part is approximately determined by the first $l-1$ moments only.

The analysis of high energy elastic scattering usually employs an absorbing disc model of the interaction. This is a model, based on the classical relation between angular momentum and the impact parameter, in which a value is assigned to each partial wave amplitude. For example, in the uniform disc, the imaginary part of the amplitudes are constant for all l up to some maximum, beyond which they are zero. The form factor of the model disc is directly connected by the relation $l = R\sqrt{\omega}$, where R is the impact parameter, to the discontinuity across the right-hand branch cut. It is possible that not all such models are consistent with unitarity and the dispersion relation which each partial wave satisfies. An attempt to investigate this problem is made in Section 4, where it is suggested that the form factor must be such that the absorption varies slowly with the impact parameter.

2. – Relation to the Hausdorff moment problem.

The invariant scattering amplitude is that defined by CHEW [1] and is related to the partial wave amplitudes by

$$A(\omega, \cos\theta) = \sum_{l=0}^{\infty} (2l+1)\, A_l(\omega)\, P_l(\cos\theta), \tag{1}$$

the partial wave amplitudes satisfying

$$A\,(\omega) = \left(\frac{\omega + M^2}{\omega}\right)^{\frac{1}{2}} \exp\,[i\delta] \sin\,\delta\,, \tag{2}$$

where δ is the complex phase shift. To obtain the limit of the imaginary part of A_l as $\omega \to +\infty$ we start from the optical theorem, which is

$$\mathrm{Im}\, A(\omega, 1) = \frac{\sigma_T}{4\pi}\sqrt{\omega(\omega + M^2)}\,. \tag{3}$$

Introducing the impact parameter R, we have from eq. (1), for a given ω,

$$\mathrm{Im}\, A(\omega, 1) = \omega R_0^2\, \mathrm{Im}\, A_l(\omega) \qquad (l \gg 1), \tag{4}$$

where it is assumed that the imaginary part of A_l is independent of l to within an order of magnitude (*), and R_0 is the maximum impact parameter. Then

$$\text{Im}\, A_l(\omega) = \sigma_T (4\pi R_0^2)^{-1} \tag{5}$$

as $\omega \to +\infty$. For the present problem, the precise behaviour of the total cross-section σ_T in this limit is unimportant. It is assumed that $\text{Im}\, A_l \to \alpha$, a constant.

A_l satisfies the dispersion relation [1,2]

$$A_l(\omega) = A_l^{\prime}(\omega) + \frac{\omega^i}{\pi}\int_0^{\infty} \mathrm{d}\omega' \, \frac{f_1(l,\omega')}{\omega'^i(\omega' - \omega - i\varepsilon)} + \frac{\omega^i}{\pi}\int_{-\infty}^{-\mu^2} \mathrm{d}\omega' \, \frac{f_2(l,\omega')}{\omega'^i(\omega' - \omega)}\,, \tag{6}$$

where f_1 and f_2 are given by

$$\begin{cases} f_1 = \lim\limits_{\varepsilon \to +0} \dfrac{1}{2i}\left[A_l(\omega' + i\varepsilon) - A\,(\omega' - i\varepsilon)\right], \\ f_2 = f_1 - \lim\limits_{\varepsilon \to +0} \dfrac{1}{2i}\left[A_l^{\prime}(\omega' + i\varepsilon) - A_l^{\prime}(\omega' - i\varepsilon)\right], \end{cases} \tag{7}$$

and i is an integer such that $1 \leqslant i \leqslant l$. The single pion exchange amplitude $A_l^{\prime}$ has only the left-hand branch cut and may therefore be separated as in eq. (6).

From (6) and the binomial expansion with exact remainder,

$$\frac{1}{\omega' - \omega} = \frac{1}{\omega'} + \frac{\omega}{\omega'^2} + \ldots + \left(\frac{\omega}{\omega'}\right)^{l-1} \frac{1}{\omega' - \omega}\,, \tag{8}$$

we obtain the set of relations

$$\int_0^{\infty} \omega'^{-i-2} f_1(l,\omega')\,\mathrm{d}\omega' = -\int_{-\infty}^{-\mu^2} \omega'^{-i-2} f_2(l,\omega')\,\mathrm{d}\omega'\,, \tag{9}$$

for integers i such that $0 \leqslant i \leqslant l-2$. In deriving (9), the property that $A_l \propto \omega^l$ near $\omega = +0$ has been used. The pole in the integrand at $\omega + i\varepsilon$ gives terms $O(\omega/\omega_0)^{2l}$, where $l = R_0\sqrt{\omega_0}$, which may be neglected.

Define

$$z' = -\mu^2/\omega'\,, \qquad \varphi(z') \equiv f_2(l,\omega') \tag{10}$$

(*) This would be exactly true for a uniform disc model.

then

$$\int_0^1 z'^i \varphi(z')\,\mathrm{d}z' = \nu_i\,, \tag{11}$$

where

$$\nu_i = (-\mu^2)^{i+1}\int_0^\infty \omega'^{-i-2} f_1(l,\omega')\,\mathrm{d}\omega'\,, \tag{12}$$

so that we have the first $l-1$ equations of a Hausdorff moment problem, the signs of the first $l-1$ moments ν_i being just $(-1)^{i+1}$. The relation between the number of changes in sign of the complete Hausdorff moment sequence and the number of zeros in $\varphi(z')$ has been derived by WIDDER [3]. Let us assume that φ is such that $\int_0^{z''}\varphi(z')\,\mathrm{d}z'$ is of bounded variation, for z'' in $(0,1)$. The statement (theorem 44) is the following: if $\varphi(z')$ has n zeros in $(0,1)$ then the sequence ν_i $(i=0,1,\ldots)$ has $m\leqslant n$ changes in sign. It follows then that φ has at least $l-2$ zeros in $(0,1)$.

Nothing can be said about the positions of the zeros except that it is likely that some are very close to the branch point at $-\mu^2$. This may be seen by considering the relation between f_2 and the absorptive parts A_1 and A_2 for the other two channels [1]

$$f_2(l,\omega') = -\tfrac{1}{2}\int_{-1-(2\mu^2/\omega')}^{1}\mathrm{d}x\,P_l(x)\,A_1(x,\omega') - \tfrac{1}{2}\int_{-1}^{1+(2M^2/\omega')}\mathrm{d}x P_l(x)\,A_2(x,\omega')\,, \tag{13}$$

where $x=\cos\theta$. The zeros x_q in the Legendre polynomials $P_l(x)$ are also required. For large l we have, by asymptotic expansion [4],

$$P_l(x) = \frac{2}{\sqrt{2\sin\theta}}\frac{(2l-1)!!}{(2l)!!}\cos\left[\frac{\pi}{4} - (l+\tfrac{1}{2})\theta\right] + O(l^{-1})\,, \tag{14}$$

with $0<\theta<\pi$. The first two zeros are

$$\begin{cases} x_1 = 1 - \dfrac{2.8}{(l+\frac{1}{2})^2} + O(l^{-4}) - \ldots\,, \\ x_2 = 1 - \dfrac{15.1}{(l+\frac{1}{2})^2} + O(l^{-4}) - \ldots\,. \end{cases} \tag{15}$$

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[3] D. V. WIDDER: *Trans. Amer. Math. Soc.*, **36**, 107 (1934).

[4] R. COURANT and D. HILBERT: *Methods of mathematical physics*, Vol. I, (New York, 1953) p. 533.

For small ω', only A_1 will contribute to f_2. The values of ω' for which the lower limit of integration in (13) coincides with a zero of $P_l(x)$ are

$$\omega_q' = -\frac{\mu^2}{2}(3 - x_q), \tag{16}$$

for $q \ll l$. In (13), for $q = 1$ and 2, the integration is over only a small area of the $x\omega'$ plane so that it is almost certainly true that A_1 is nearly constant and therefore that the first zero in $f_2(l, \omega')$ will lie between ω_1' and ω_2'. It is possible that the positions of other zeros may be given by the same arguments. It is a further consequence of the rapid oscillation of $P_l(x)$ that f_2 will be small near $-\mu^2$. This part of the unphysical branch cut, which may include many zeros, is therefore unimportant in eq. (6).

3. – Methods of solution.

The contribution from the unphysical branch cut is, for $i = 1$ in eq. (6),

$$B(z) = \frac{1}{\pi}\int_0^1 \frac{q(z')\,\mathrm{d}z'}{z + z'}, \tag{17}$$

where $z = \mu^2/\omega$. We indicate three methods of calculating $B(z)$ in terms of the moments, which involve representing $(z+z')^{-1}$ by a polynomial in z'.

A) The most simple method is a binomial expansion. This gives

$$B(z) = \frac{1}{\pi}\left[\frac{\nu_0}{z} - \frac{\nu_1}{z^2} + \ldots + (-1)^{l-2}\frac{\nu_{l-2}}{z^{l-1}} + R(z)\right], \tag{18}$$

where $R(z)$ represents the higher moments which are not given.

B) An expansion of $(z+z')^{-1}$ in orthogonal polynomials, for example, those defined ([5]) in $(0, 1)$ by

$$L_i(z') = \frac{1}{(i)!}\frac{\mathrm{d}^i}{\mathrm{d}z'^i}[z'^i(1 - z')^i]. \tag{19}$$

([5]) J. A. Shohat and J. D. Tamarkin: *The problem of moments* (*Amer. Math. Soc.*, 1950) p. 90.

The coefficients in the expansion are obtained by the orthogonality relation

$$(20) \qquad (2i+1)\int_0^1 L_i(z')\,L_j(z')\,\mathrm{d}z' = \delta_{ij}\,,$$

C) Define the Laplace transform $W(z, u)$ of $(z+z')^{-1}$ so that

$$(21) \qquad B(z) = \frac{1}{\pi}\int_0^\infty \mathrm{d}u \int_0^1 \mathrm{d}z'\, W(z, u)\varphi(z') \exp\left[-z'u\right].$$

The upper limit in the u integration may be replaced by a finite number, u_0, the remainder being represented by an expansion in the polynomials of method *B*).

4. – The imaginary part of the amplitude.

The methods of solution given in Section **3** can be used to investigate the real part of A_l for a given $f_1(l, \omega')$. The total scattering cross-section given by single pion exchange is proportional to ω^{-2}, as $\omega \to +\infty$. For large ω, the single pion exchange amplitude A_l' is therefore a small proportion of the complete amplitude and it is sufficient to calculate the difference $A_l'' = A_l - A_l'$.

The amplitude should satisfy the following three conditions:

i) A_l satisfies the dispersion relation (6),

ii) $A_l \propto \omega^l$ for ω near $+0$,

iii) $|A_l| \leqslant \left((\omega+M^2)/\omega\right)^{\frac{1}{2}}$ for $\omega > 0$.

The problem attempted in this section is to determine some of the properties that f_1 must have in order that the amplitude A_l satisfies these three conditions. No complete or rigorous solution is achieved but it appears to be possible to construct model (*) functions f_1 for which A_l does not satisfy condition iii).

The method is to replace $(z+z')^{-1}$ by a polynomial in z'. The most convergent polynomial is obtained by method *C*). The remainder (corresponding to $u > u_0$) is $(z+z')^{-1}\exp\left[-\left(1+(z'/z)\right)\right]$ where $u_0 = 1/z$. This function is small for $z' \gg z$ but is important. Owing to the exponential term, $(z+z')^{-1}$ need be represented by a polynomial only for a small range $0 < z' < z/c$, where c is a

(*) These calculations were carried out on the Ferranti Mercury digital computer at the University of Oxford Computing Laboratory.

constant less than unity. Introduce the variable y, $0 < y < 1$, so that

$$(z + z')^{-1} = \frac{c}{z}(c + y)^{-1},$$

then $(c+y)^{-1}$ may be represented by an expansion in the orthogonal polynomials (19).

Let us consider neutron-proton scattering at 25 GeV, so that representative numerical values are $l = 20$ and $\omega_0 = 205\mu^2$. For a model

$$f_1 = 0\,, \qquad \omega' < \omega_0$$

$$f_1 = \alpha\,, \qquad \omega' > \omega_0$$

the calculated values of A_l'' are shown in Fig. 1. The sharp negative increase with ω is inconsistent with condition iii) unless $\alpha \ll 0.5$. For these values of ω, the ratio of terms containing ν_{l-2} and ν_0 is very small and the convergence of the method is good for the first $l-1$ moments.

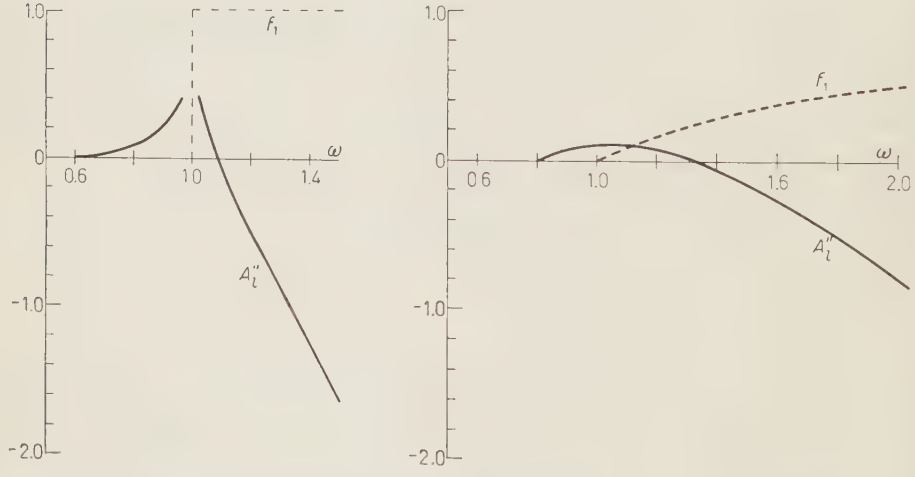

Fig. 1. – The amplitudes $A_l''(\omega)$ and $f_1(l, \omega)$ are in units of α. ω is in units of ω_0.

The rate of negative increase clearly depends on the rate of decrease of the moments ν_i. This is demonstrated by the second part of Fig. 1, in which $f_1 \propto (1 - (\omega_0/\omega))$ for $\omega > \omega_0$. For this model, the negative increase is less rapid. In order that condition iii) be satisfied, f_1 must be such that with increasing ω, the higher moments become important before the intermediate moments, for example $3 < i \leqslant l-2$. The moments for $0 \leqslant i \leqslant l-2$ must therefore decrease as rapidly as possible with increasing i. This implies a function f_1 equivalent

to a model disc having a diffuse form factor, so that the absorption varies slowly with the impact parameter.

For $\omega > \omega_0$, both the real part of the phase shift and its momentum derivative are negative. The idea that the rate of negative increase of A_l'' must be limited is therefore accordant with the lower bound for the momentum derivative of a real phase shift derived by WIGNER (6). In the present case though, the physical interpretation is not so clear.

(6) E. P. WIGNER: *Phys. Rev.*, **98**, 145 (1955).

RIASSUNTO (*)

Si dimostra che la forma dell'ampiezza a bassa energia determina $l-1$ momenti della funzione nella relazione di dispersione dell'onda parziale che dà la discontinuità nel taglio attraverso il ramo non fisico. Si dà una disuguaglianza per il numero degli zeri di questa funzione. Si studia la forma della successione di momenti necessaria a dare un'ampiezza di scattering dell'onda parziale compatibile con l'unitarietà e le relazioni di dispersione dell'onda parziale. Si suggerisce che la successione debba essere tale che l'assorbimento sia una funzione lentamente variabile del parametro d'urto.

(*) *Traduzione a cura della Redazione.*

Coherent and Incoherent Nuclear π^0 Photoproduction.

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Summary. — A simple treatment of neutral pion photoproduction in complex nuclei using closure and neglecting nuclear absorption of the π^0, is considered in the region of small momentum transfer. In addition to the usual incoherent process in a nucleus the fact that the nucleon on which the π^0 is produced does not change its charge gives rise to the possibility of a coherent addition of the production amplitude over the whole nucleus provided the momentum transfer to the nucleon is not large compared to the inverse nuclear size. Because the spin independent part of the production amplitude which is responsible for the coherent process, has a $\sin\theta$ dependence (θ is the angle between pion and photon), the coherent cross-section will vanish in the forward direction and will reach a maximum at an angle $\theta_m = (1.87)\ [Kr_0A^{\frac{1}{3}}]^{-1}$, where K is the photon momentum and $r_0A^{\frac{1}{3}}$ the charge radius of the nucleus. Due to the fact that the maximum does not occur at zero angle the ratios of the coherent cross-section at θ_m will behave as $A^{\frac{1}{3}}$ rather than A^2 in different elements. An estimate of the incoherent process is included and it is shown that this cross-section is less than the sum of A free nucleons for small momentum transfers and reaches the free nucleon value as the momentum transfer approaches the Fermi momentum.

The possibility [1] of measuring the π^0 lifetime through the Coulomb production of neutral pions [2] makes the study of other means of producing neutral pions important in giving a clear cut interpretation to the lifetime experiments. Furthermore, these other means of producing neutral pions may prove useful in gaining information about complex nuclei. For these

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[1] A. V. Tollestrup, S. M. Berman, R. Gomes and H. Ruderman: *Proceedings of the Rochester Conference* (New York, 1960), p. 27.

[2] H. Primakoff: *Phys. Rev.*, **81**, 899 (1951).

reasons we consider here an elementary treatment of π^0 production in complex nuclei. The approach used is similar to methods used by LAX and FESHBACH [3].

The general matrix element T for the production of neutral pions by photons on a target nucleon can can be written as

$$(1) \qquad T = \exp[i\boldsymbol{q}\cdot\boldsymbol{r}][(B+\boldsymbol{\sigma}\cdot\boldsymbol{C})\tau^-\tau^+ + (D+\boldsymbol{\sigma}\cdot\boldsymbol{E})\tau^+\tau^-] = \\ = \exp[i\boldsymbol{q}\cdot\boldsymbol{r}][a\tau^-\tau^+ + b\tau^+\tau^-],$$

where $\boldsymbol{q} = \boldsymbol{K} - \boldsymbol{p}$ is the difference between the photon and pion three-momenta and $\tau^{+,-} = (\tau_x \pm i\tau_y)/2$ are the isospin operators for nucleons. The cross-sections for neutrons and protons are proportional to $|a|^2$ and $|b|^2$ respectively. For a nucleus with A nucleons the matrix element will be of the form of a sum over T_i for each nucleon

$$(2) \qquad \langle F|M|I\rangle = \langle F|\sum_{i=1}^{A}\exp[i\boldsymbol{q}\cdot\boldsymbol{r}_i][a_i\tau_i^-\tau_i^+ + b_i\tau_i^+\tau_i^-]|I\rangle,$$

where $|I\rangle$ and $|F\rangle$ refer to the initial and final states of the nucleus.

The quantities B, C, D, E will, in the case of a single nucleon, be functions of the various momenta. If we consider the system in which the target nucleon is at rest these quantities will depend only on p and k and not on the nucleon momenta which can be eliminated by the conservation of momentum. In the treatment here the assumption is made that in order to connect the single particle cross-sections with the cross-section in complex nuclei the quantities $B \ldots E$ shall be approximately independent of nucleon momentum.

The differential cross-section is obtained by squaring the matrix element given by eq. (2) and summing over all final states F which are energetically compatible with the condition that

$$E_F = K - (p^2 + m^2)^{\frac{1}{2}}.$$

Since in medium to heavy nuclei many states are contained in an energy interval of about 10 MeV and since there is probably that much spread in the experimental determination of the π^0 energy, we make the approximation that the sum over final states can be replaced by closure and that for reasonably high energy pions ($E > 700$ MeV) the pion energy is approximately equal to the photon energy. In that case

$$(3) \qquad \sum_F |\langle F|M|I\rangle|^2 = \langle I|\sum_i\sum_j \exp[i\boldsymbol{q}\cdot(\boldsymbol{r}_i - \boldsymbol{r}_j)]\cdot \\ \cdot[\tau_i^-\tau_i^+a_i + \tau_i^+\tau_i^-b_i][\psi_j^-\psi_j^+a_j^* + \tau_j^+\tau_j^-b_j^*]|I\rangle.$$

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[3] M. LAX and H. FESHBACH: *Phys. Rev.*, **81**, 189 (1951).

Following LAX and FESHBACH [3] we divide the double sum into two parts where $i=j$ and $i\neq j$. The case $i=j$ is quite simply

$$\langle I|\sum_i T_i T_i^*|I\rangle = N(|B|^2+3|C|^2)+Z(|D|^2+3|E|^2)\,, \tag{4}$$

where N and Z are the number of neutrons and protons respectively. To proceed with the case $i=j$ it is necessary to make two assumptions about the nuclear system namely that (a) there are only two nucleon-nucleon correlation functions, one for space symmetric motion and one for space anti-symmetric motion and (b) that these functions depend only on the distance between particles not on direction. Thus assumption (b) means that we are neglecting any surface effects on the nucleus.

Taking advantage of assumption (b) we can write the terms in $i\neq j$ as

$$\sum_{i,j}{}' \langle I|T_i T_j^*|I\rangle = \sum_{i,j}{}' \langle I|V_{ij}O_{ij}|I\rangle\,,$$

or

$$\begin{aligned}\sum_{i,j}\langle I|T_iT_j^*|I\rangle = \langle I|\sum_{i,j}\frac{\sin qr_{ij}}{qr_{ij}}[&\tau_i^-\tau_i^+\tau_j^-\tau_j^+(|B|^2+|C|^2\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j)+\\ &+\tau_i^-\tau_i^+\tau_j^+\tau_j^-(BD^*+CE^*\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j)+\tau_i^+\tau_i^-\tau_j^-\tau_j^+(B^*D+C^*E\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j)+\\ &+\tau_i^+\tau_i^-\tau_j^+\tau_j^-(|D|^2+|E|^2\boldsymbol{\sigma}_j\cdot\boldsymbol{\sigma}_j)]|I\rangle\,,\end{aligned} \tag{5}$$

where $\sum'$ means $i\neq j$ and V_{ij} is the part of $T_iT_j^*$ which depends on space. Following reference [3] we write eq. (5) as

$$\sum_{i,j}{}'\langle I|T_iT_j^*|I\rangle - \sum_{i,j}{}'\tfrac{1}{2}(V_s+V_a)\langle I|O_{ij}|I\rangle+\tfrac{1}{2}(V_s-V_a)\langle I|O_{ij}P_{ij}|I\rangle\,, \tag{6}$$

where P_{ij} is the space exchange operator

$$P_{ij}=-\tfrac{1}{4}(1+\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j)(1+\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j)\,,$$

and where $V_{s,a}$ are given in terms of integrals over the space symmetric and space anti-symmetric correlation functions F_s, F_a as

$$V_{s,a}=\int\frac{\sin q|\boldsymbol{r}-\boldsymbol{r}'|}{q|\boldsymbol{r}\cdot\boldsymbol{r}'|}F_{s,a}(|\boldsymbol{r}-\boldsymbol{r}'|)\,\mathrm{d}^3r\,\mathrm{d}^3r'\,. \tag{7}$$

The spin and isotopic spin matrix elements can be evaluated by methods

similar to WIGNER (4) or PRIMAKOFF (5), for example

$$\langle I|\sum_{i,j}{}' \tau_i^+\tau_i^-\tau_j^+\tau_j^-(1+\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j)|I\rangle = 2Z^2 .$$

Evaluating the other terms in a similar way we have that

$$\sum_{i,j}\langle I|O_{ij}|I\rangle = N^2|B|^2+Z^2|D|^2+NZ(B^*D+D^*B) -$$
$$-N(|B|^2+3|C|^2)-Z(|D|^2+3|E|^2)+S_n^2+S_p^2 ,$$

$$\sum_{i,j}{}'\langle I|O_{ij}P_{ij}|I\rangle = -(N^2/2)(|B|^2+3|C|^2)-(Z^2/2)(|D|^2+3|E|^2)-$$
$$-[(E^*C+C^*E)+(B^*D+D^*B)]\left[A-2\left(T^2-(T^3)^2\right)\right]+$$
$$+\tfrac{1}{2}(E^*C+C^*E)\sum_{i,j}\tau_i^-\tau_j^+\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j+\tfrac{1}{2}S_n^2+\tfrac{1}{2}S_p^2 ,$$

where $S_{p,n}=\langle I|\sum_i[(1\pm\tau_i^3)/2]\sigma_i|I\rangle$ and where T and T^3 are isotopic spin and third component of isotopic spin of the nucleus, respectively.

Neglecting terms of order unity in (V_s+V_a) and of order A in (V_s-V_a) we have the differential cross-section in a medium to heavy nucleus as

$$\text{(8)}\qquad \mathrm{d}\sigma/\mathrm{d}\Omega = [N^2|B|^2+Z^2|D|^2+NZ(B^*D+D^*B)](V_s+V_a)/2+$$
$$+[N(|B|^2+3|C|^2)+Z(|D|^2+3|E|^2]\{1-(V_s+V_a)/2\}-$$
$$-[N^2(|B|^2+3|C|^2)+Z^2(|D|^2+3|E|^2)](V_s-V_a)/4 .$$

To the extent that the π^0 photoproduction cross-section from neutrons is the same as protons we would have $B=D$, $C=E$ and the differential cross-section from a nucleus would take the simple form

$$\text{(9)}\qquad \mathrm{d}\sigma/\mathrm{d}\Omega = A^2|B|^2(V_s+V_a)/2+A\,\mathrm{d}\sigma_{\mathrm{H}}/\mathrm{d}\Omega\{1-(V_s+V_a)/2-$$
$$-[A-2NZ/A](V_s-V_a)/4\} = \mathrm{d}\sigma_c/\mathrm{d}\Omega+\mathrm{d}\sigma_I/\mathrm{d}\Omega ,$$

where $\mathrm{d}\sigma_{\mathrm{H}}/\mathrm{d}\Omega$ is the cross-section in hydrogen and where $\mathrm{d}\sigma_c$ is the coherent part of the cross-section, *i.e.*, proportional to A^2 and $\mathrm{d}\sigma_I$ is the incoherent part proportional to A.

Because the correlation functions are normalized to unity we see that the incoherent part of the cross-section vanishes at $q^2=0$ and only the coherent

(4) E. P. WIGNER: *Phys. Rev.*, **56**, 519 (1939).
(5) H. PRIMAKOFF: *Rev. Mod. Phys.*, **31**, 3 (1959).

part apparently remains. However, because B has a $\sin\theta$ dependence, as shown below, the coherent part will also vanish at $q^2 = 0$.

In order to estimate the angular shape of both the incoherent and coherent parts it is necessary to use some specific form for the nucleon-nucleon correlation functions. For this purpose we take the correlation functions used by PRIMAKOFF (5) which have given good agreement for the case of muon capture, namely,

$$F_{s,a}(|\boldsymbol{r}-\boldsymbol{r}'|) = H[\varrho(r)\varrho(r') \pm f(|\boldsymbol{r}-\boldsymbol{r}'|)], \tag{10}$$

where

$$f(|\boldsymbol{r}-\boldsymbol{r}'|) \begin{cases} = 1 & |\boldsymbol{r}-\boldsymbol{r}'| < d, \\ = 0 & |\boldsymbol{r}-\boldsymbol{r}'| > d, \end{cases}$$

where H is a normalization factor and where $\varrho(r)$ is the ground state expectation value of the nucleon density.

The length d which plays an important role for PRIMAKOFF enters here only in the incoherent part in an exactly similar manner.

Normalizing the correlation functions $F_{s,a}$ to unity and for simplicity taking constant density we have from eq. (10) that

$$(V_s + V_a)/2 = [3/R^3q^3]^2[\sin qR - qR\cos qR]^2, \tag{11}$$

where R is the nuclear radius.

From this form of $V_s + V_a$ the coherent part of the cross-section will vanish (6) when $\operatorname{tg} qR = qR$, or using the approximate relation $q \approx K\theta$ for small angles, will vanish at an angle

$$\theta_0 = 4.6/(Kr_0A^{\frac{1}{3}}), \tag{12}$$

where $r_0 = 1.25 \cdot 10^{-13}$ cm.

Even though $(V_s + V_a)$ is maximum at $\theta = 0°$ the coherent part will have its maximum between 0 and θ_0 due to the fact that the spin independent amplitude B must have the form

$$B \sim \boldsymbol{e}\cdot\boldsymbol{K}\times\boldsymbol{p},$$

for a target nucleon at rest and with $\boldsymbol{e}$ the photon polarization. Averaging over polarizations makes B proportional to $\sin\theta$ and hence for small angles

(6) K. BERKELMAN and J. WAGGONER: *Phys. Rev.*, **117**, 1364 (1960).

the coherent differential cross-section will be proportional to

$$A^2[3/R^3K^3]^2[(1/\theta)(\sin KR\theta - KR\theta \cos KR\theta)]^2 ,$$

where we have used $q \approx KR\theta$ and have a maximum at

$$\theta_m = (1.87)/(Kr_0A^{\frac{1}{3}}) . \tag{13}$$

For example, in Pb at $K=1$ GeV θ_m occurs at approximately 3°.

As a consequence of θ_m occurring at a non-zero angle we see that substituting θ_m in $d\sigma_c$ the maximum values of the coherent differential cross-section vary as $A^{\frac{4}{3}}$ rather than A^2. Furthermore integrating between 0 and θ_0 yields that the total coherent cross-section varies as $A^{\frac{2}{3}}$.

In order to have an absolute number for the coherent cross-section some knowledge of B would be required but because a forward direction experiment in hydrogen can only give C and because there is no reliable theoretical estimate for B at energies above 500 MeV we are unable to give a definite value for σ_c.

In determining the nuclear cross-section we have performed the sum over all final states but as far as the coherent part is concerned it is readily seen that only the ground state can give a contribution proportional to A^2. This ground state matrix element is simply the electromagnetic form factor of the nucleus. Therefore instead of eq. (11) for V_s+V_a we could have used the electromagnetic form factor as given directly by experiments on electron scattering. This would give essentially the same results as above since these are quite insensitive to the details of the various nuclear form factors.

The incoherent part of the cross-section can be readily evaluated in terms of the parameter d introduced by PRIMAKOFF [5]. From eq. (10) we have

$$\begin{aligned} d\sigma_I/d\Omega = A(d\sigma_H/d\Omega)\,[1 - (3/R^3q^3)^2\{1 - (d^3/2r_0^3)(1 - 2NZ/A^2)\}\cdot \\ \cdot(\sin qR - qR\cos qR)^2 - (3/2r_0^3q^3)(1 - 2NZ/A^2)(\sin qd - qd\cos qd)] . \end{aligned} \tag{14}$$

This can be simplified for small angles, *i.e.* for $qR > 1$ as the terms in qR will then be negligible. In that case

$$d\sigma_I/d\Omega = A(d\sigma_H/d\Omega)[1 - \tfrac{3}{4}(1/r_0^3q^3)(\sin qd - qd\cos qd)] , \tag{15}$$

where $2NZ/A^2$ has been approximated by $\frac{1}{2}$ which is good to a few percent even for lead.

The term in brackets in (15) is just the Primakoff factor for the nuclear effect in muon capture and arises here in a similar way as a result of corre-

lations between nucleons. These correlations make the cross-section in a nucleus smaller than the sum of A individual nucleons. As the momentum transfer approaches the Fermi momentum the nucleon acts more and more like a free particle and the cross-section approaches the value of the sum of A individual particles as may be seen from the behaviour of eq. (15).

In the region $qd < 1$, by expanding (15) and using Primakoff's estimate of d/r_0, namely that

$$(d/r_0)^3 \approx 3 ,$$

we have that

$$\mathrm{d}\sigma_I/\mathrm{d}\Omega = (A/4)(\mathrm{d}\sigma_\mathrm{H}/\mathrm{d}\Omega) ; \qquad 1/KR < \theta < m_\mu/K ,$$

or using the values of $\mathrm{d}\sigma_\mathrm{H}$ for small θ obtained by BERKELMAN and WAGGONER ([8])

$$\mathrm{d}\sigma_I/\mathrm{d}\sigma_\mathrm{H} \approx (0.25 \cdot 10^{-30}) A ; \qquad 1/KR < \theta < m_\mu/K ,$$

where m_μ is the muon mass.

It is interesting to compare $\mathrm{d}\sigma_I$ with $\mathrm{d}\sigma_C$ in the angular range that is studied for the Coulomb production of neutral pions: for photon energies around 1 GeV this region is in the neighbourhood of 0.01 rad. From (14) and (9) we have in this region where $qR < 1$ that

$$\mathrm{d}\sigma_I/\mathrm{d}\Omega = \left(\frac{\mathrm{d}\sigma_\mathrm{H}}{\mathrm{d}\Omega}\right) A q^2 R^2/20 , \tag{16}$$

where we have used the value $(d/r_0)^3 \approx 3$. The momentum transfer q is determined from the relation

$$q^2 = p^2 + K^2 - 2p \cos\theta ,$$

where the pion momentum p is given by

$$p^2 = (K - \delta)^2 - m_\pi^2 ,$$

and where δ is the average excitation energy of the nucleus. For small θ the momentum transfer is approximately given by

$$q^2 = (\delta + m_\pi^2/2K)^2 + (K\theta)^2 . \tag{17}$$

The ratio of incoehrent to coherent cross-section in this region is then

$$\mathrm{d}\sigma_I/\mathrm{d}\sigma_C = [3\,|\,C\,|^2/|\,B_0\,|^2][q r_0^2/20\,\theta^2] A^{-\frac{1}{3}} , \tag{18}$$

where $B_0 = B/\sin\theta$.

For example in lead and carbon at $K=1$ GeV

$$\mathrm{d}\sigma_I/\mathrm{d}\sigma_C = \frac{3|C|^2}{|B_0|^2}\begin{cases}2.5\text{ Pb}\\ 6.5\text{ C}\end{cases},$$

where we have assumed that δ is approximately 10 MeV (7).

The ratio of incoherent to coherent cross-sections for very small angles has also been calculated by CHIUDERI and MORPURGO (8). With the assumption that $B_0 = C$ these authors find a value in lead which is about 50% smaller than that given here in rough agreement; however, our dependence on mass number A for eq. (18) is very different from that of reference (8). For aluminium, eq. (18) gives the value 15 where the above authors have a value 40. These differences are probably due to the various approximations used in reference (8) such as neglect of the two particle contribution and higher moments in the expansion of $\exp[iq\cdot r]$.

* * *

The author would like to thank Dr. J. S. BELL for a helpful discussion and gratefully acknowledges the hospitality of CERN.

(7) A similar expression to that of eq. (14) is obtained for the nuclear photoproduction of π^+ mesons. In that case we have

$$(\mathrm{d}\sigma/\mathrm{d}\Omega)_{\pi^+} = (\mathrm{d}\sigma/\mathrm{d}\Omega)_{\mathrm{H},\pi^+} Z\{1-(3/R^3q^3)^2[1-(\mathrm{d}^3/2r_0)(1-Z/A)]\cdot$$
$$\cdot(\sin qR - qR\cos qR)^2 - (3/2r_0^3q^3)(1-Z/A)(\sin qd - qd\cos qd)\}\ .$$

Using the value of the hydrogen cross-section given by DIXON and WALKER (F. D. DIXON and R. W. WALKER: *Proceedings of Rochester Conference* (New York, 1960)), we have from the above equation that for $qR \leqslant 1$ the value of the π^+ cross-section in Pb is approximately 16 μb per sr.

(8) C. CHIUDERI and G. MORPURGO: *Nuovo Cimento*, **19**, 497 (1961).

RIASSUNTO (*)

Prendo in considerazione un semplice trattamento della fotoproduzione di pioni neutri in nuclei complessi usando la chiusura e trascurando l'assorbimento nucleare del π^0, nella zona di piccolo trasferimento di impulso. In aggiunta all'usuale processo incoerente, il fatto che il nucleone sul quale si produce il π^0 non cambi la sua carica, dà adito alla possibilità di un'aggiunta coerente dell'ampiezza di proiezione su tutto il nucleo, purchè il trasferimento di impulso non sia grande in confronto all'inverso della grandezza del nucleo. Poichè la parte dell'ampiezza di produzione indipendente dallo spin, che è responsabile del processo coerente, dipende da $\sin\theta$ (θ è l'angolo fra il pione ed il fotone), la sezione d'urto coerente si annulla nella direzione anteriore e raggiunge un massimo per un angolo $\theta_m = (1.87)[Kr_0A^{\frac{1}{3}}]^{-1}$, in cui K è l'impulso del fotone e $r_0A^{\frac{1}{3}}$ il raggio di carica del nucleo. Per il fatto che il massimo non ha luogo ad un angolo di 0, i rapporti della sezione d'urto coerente a θ_m varieranno proporzionalmente ad $A^{\frac{1}{3}}$ piuttosto che ad A^2 in differenti elementi. Includo una valutazione del processo incoerente e mostro che questa sezione d'urto è inferiore alla somma di A nucleoni liberi per piccolo trasferimento di impulso e raggiunge il valore del nucleone libero quando il trasferimento d'impulso si avvicina all'impulso di Fermi.

(*) *Traduzione a cura della Redazione.*

Off-Shell Pion-Nucleon Scattering and Dispersion Relations.

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Summary. — The problem of pion-nucleon scattering when one of the pions is virtual is discussed. By assuming that one-dimensional dispersion relations in the energy variable can be written for suitably defined invariant amplitudes, and that also in the off-shell case the dispersive integrals are dominated by the $T=J=\frac{3}{2}$ amplitude, an Omnès-type integral equation is deduced for this amplitude. The rigorous solution of this equation is discussed and reduced to an approximate simpler form which is more suitable for numerical calculations. This form is particularly simple for squared four-momenta of the virtual pion less than ~ 5 squared pion masses.

1. – Introduction.

The analytic continuation of a scattering amplitude, when one of the incoming particles becomes virtual, has a certain interest in connection with the description of inelastic processes in terms of a single particle exchange.

It has been shown by Da Prato [1] that the most naïve formulation of this interaction picture which has been applied so far [2,3] works satisfactorily

[1] G. Da Prato: to be published. See also ref. [2] where, however, the absolute normalization is wrong by a factor $\frac{1}{2}$ (F. Salzman, A. Stanghellini and G. C. Wick: private communications).

[2] F. Selleri: *Phys. Rev. Lett.*, **6**, 64 (1961).

[3] E. Ferrari: *Nuovo Cimento*, **15**, 652 (1960) and *Phys. Rev.*, **120**, 988 (1960); F. Salzman and G. Salzman: *Phys. Rev.*, **120**, 599 (1960) and *Phys. Rev. Lett.*, **5**, 377 (1960); S. D. Drell: *Phys. Rev. Lett.*, **5**, 278, 342 (1960); J. Iizuka and A. Klein: *Phis. Rev.*, **123** 669 (1961).

well in the process of single pion production in proton-proton collisions at 2.85 GeV (4). At this energy the experimentally observed low- and high-energy peaks in the lab. spectra of the final nucleons are theoretically reproduced: these peaks occur for low values of the squared four-momentum Δ^2 of the virtual intermediate particle (5). For higher values of Δ^2 the agreement with experiment becomes worse: in this region, however, the theoretical calculations cannot be trusted because the approximations involved are no longer justified. We do not know whether the discrepancies between the predictions of the peripheral model and the experiment at high Δ^2 are mainly caused by the physical occurrence of other mechanisms of interaction (many-particle exchange) or are the consequence of the assumption that the off-shell 3.3 amplitude still dominates the off-shell scattering and is equal to its on-shell value. The hypothesis that the low-energy off-shell scattering is still dominated by the 3.3 resonance can be phenomenologically justified from the fact that in the quoted experiment at 2.85 GeV, there is evidence at the same time of a very important one-pion exchange contribution and of the formation of the 3.3 isobar. The purpose of this paper is to investigate the analytic continuation of the 3.3 partial wave amplitude to unphysical values of the mass of the incoming pion. By assuming one-dimensional dispersion relations for « off-shell » invariant amplitudes suitably chosen, we shall derive a linear integral equation for the « off-shell » partial amplitude of the $T = J = \frac{3}{2}$ state by closely following the procedure by CHEW, GOLDBERGER, LOW and NAMBU (6). This equation can be reduced to an equation of the type already discussed and solved by OMNÈS (7). Its solution will give us the correct energy and Δ^2-dependence of the off-shell 3.3 amplitude.

In Sect. **2** we discuss the so-called pole approximation, which has been used so far in the picture of the production processes through the exchange of a single particle, and we point out the assumptions which are implied in it. Sect. **3** contains the definition of the invariant and partial wave amplitudes for the off-shell scattering as well as certain relations similar to the unitarity condition in the on-shell case. In Sect. **4** the dispersion relations for the off-shell amplitudes are written down and an equation is derived for the $T = J = \frac{3}{2}$ off-shell amplitude, by using the same procedure as in Ref. (6).

In Sect. **5** it is shown that the « unitarity » condition allows this equation to be transformed into a linear integral equation of the Omnès type which in

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(4) G. A. SMITH, H. COURANT, E. FOWLER, H. KRAYBILL, J. SANDWEISS and H. TAFT: University of Yale, preprint.

(5) $\Delta^2 = k^2$, where k is the 4-momentum of the exchanged pion. We use the metric $k^2 = \boldsymbol{k}^2 - k_0^2$.

(6) G. F. CHEW, M. L. GOLDBERGER, F. E. LOW and Y. NAMBU: *Phys. Rev.*, **106**, 1337 (1957), hereafter referred to as CGLN.

(7) R. OMNÈS: *Nuovo Cimento*, **8**, 316 (1959).

principle can be solved exactly. By performing the static limit on this equation, another approximate equation is derived. Finally, in Sect. 6 the solution of the latter equation is reduced to a simpler form which is more suitable for numerical calculations. For $\Delta^2 \lesssim 5\mu^2$ this approximate solutioni s proportional to the on-shell 3.3 amplitude. For higher Δ^2 the dominant part of the solution is still proportional to the physical amplitude.

2. – The one-particle exchange interaction picture.

We briefly recall in this section the procedure which has been used so far [1-3] in the calculation of single-particle exchange effects. If we consider in general the process (the same simbol is used for a particle and its four-momentum)

$$p_1 + p_2 \to q_1 + q_2 + \ldots + q_n \,, \tag{1}$$

we can define an invariant M-matrix element from the S-matrix element through

$$S_{fi} = \delta_{fi} - i(2\pi)^{4-\frac{3}{2}(n+2)} 2^{-b/2} \Big[\prod_{i=1}^{f} m_i \Big]^{\frac{1}{2}} \cdot \cdot [p_{10} p_{20} q_{10} \ldots q_{n0}]^{-\frac{1}{2}} \delta^4(q_1 + \ldots + q_n - p_1 - p_2) M_{fi} \,, \tag{2}$$

where $b(f)$ is the total number of bosons (fermions) present in the initial and final states, $p_{10} \ldots q_{n0}$ are the energy components of the 4-vectors $p_1 \ldots q_n$ respectively. For the process of single pion production in nucleon-nucleon collisions

$$p_1 + p_2 \to q_1 + q_2 + q \,, \tag{3}$$

(p_i and q_i nucleons, q pion of charge index β) supposed to occur through the exchange of a single pion of 4-momentum $k = p_2 - q_2$ and of charge index α (Fig. 1), we have

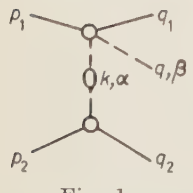

Fig. 1.

$$M_{fi}^{\beta} = \frac{\overline{u}(q_2) G_r \gamma_5 \tau_\alpha u(p_2)}{\Delta^2 + \mu^2} K(\Delta^2) K'(\Delta^2) M_{\pi N}^{\alpha\beta}(\omega^2, t^2, \Delta^2) \,. \tag{4}$$

In eq. (4) $\Delta^2 = k^2 = (q_2 - p_2)^2$, $\omega^2 = -(q_1 + q)^2$, $t^2 = (q_1 - p_1)^2$; τ_α are isospin matrices for the nucleons; G_r is the renormalized and rationalized pion-nucleon coupling constant: $K(\Delta^2)$ is the « pionic

form factor » of the nucleon, defined by

$$\langle q_2 | j_\alpha(0) | p_2 \rangle = (2\pi)^{-3} \left[\frac{m^2}{p_{20} q_{20}}\right]^{\frac{1}{2}} \bar{u}(q_2) G_r \gamma_5 \tau_\alpha u(p_2) K(\Delta^2) , \tag{5}$$

(where $j_\alpha(0)$ is the current of the pion field and m the nucleon mass) $i(2\pi)^{-4} K'(\Delta^2)(\Delta^2+\mu^2)^{-1}$ is the complete pion propagator; $M^{\alpha\beta}_{\pi\mathcal{N}}$ is defined through

$$\langle q_1, q\beta | j_\alpha(0) | p_1 \rangle = (2\pi)^{-\frac{3}{2}} \left[\frac{m^2}{2 q_0 q_{10} p_{10}}\right]^{\frac{1}{2}} M^{\alpha\beta}_{\pi\mathcal{N}}(\omega^2, t^2, \Delta^2), \tag{6}$$

The latter quantity for $\Delta^2 = -\mu^2$ reduces to the M-matrix element for the physical $\pi\mathcal{N}$ scattering

$$k + p_1 \to q + q_1 , \tag{7}$$

at total c.m. energy ω and nucleon momentum transfer t^2. At the « pole », $\Delta^2 = -\mu^2$, which corresponds to a real exchanged pion, we have $K(-\mu^2) = K'(-\mu^2) = 1$. Therefore, near the pole everything in (4) is known, and we get

$$M^{\beta(\text{pole})}_{fi} = \frac{\bar{u}(q_2) G_r \gamma_5 \tau_\alpha u(p_2)}{\Delta^2 + \mu^2} M^{\alpha\beta}_{\pi\mathcal{N}}(\omega^2, t^2) .$$

We might expect that the use of eq. (8) in the physical region for small Δ^2 ($\Delta^2 \approx \mu^2$) could give a good approximation, since the pole is not far and the Δ^2-dependence of all the off-shell quantities is supposed to be smooth. This approximation (pole approximation) would be in fact reasonable if $M_{\pi\mathcal{N}}$ contained only an s wave interaction. For higher order waves, however, the situation is not free from ambiguities.

In fact the t^2-dependence of $M_{\pi\mathcal{N}}$ occurs only through the scattering angle $\varepsilon = \widehat{\boldsymbol{q}_1 \boldsymbol{p}_1}$ in the c.m. system of q_1 and q. This angle is defined by the relation

$$\cos\varepsilon = \frac{2 p_{10} q_{10} - t^2 - 2m^2}{2 |\boldsymbol{p}_1| |\boldsymbol{q}_1|} , \tag{9}$$

and turns out to be Δ^2-dependent, since p_{10} and $|\boldsymbol{p}_1|$ depend on Δ^2. Its « on-shell » value is given by performing the limit $\Delta^2 \to -\mu^2$

$$(\cos\varepsilon)_{\text{on}} = 1 - \frac{t^2}{2 |\boldsymbol{q}_1|^2} . \tag{10}$$

Now, if one considers the allowed phase space of ω^2 and t^2 for fixed Δ^2, one finds that, also for Δ^2 small, that is in the region where the pole approximation

should be expected to hold, expression (9) is always limited between -1 and $+1$, while expression (10) reaches values considerably outside this interval, and diverges at threshold. We then see that one cannot keep the whole element $M^{\alpha\beta}_{\pi N}(\omega^2, t^2, \Delta^2)$ on-shell, since the angular part contained in it depends very strongly on Δ^2, also for small Δ^2. The most reasonable modification of the « pole approximation » consists in writing explicitly the angular dependence of $M^{\alpha\beta}_{\pi N}$, using for $\cos\varepsilon$ the Δ^2-dependent expression (9) and treating the partial wave amplitudes as Δ^2-independent. This approximation has been used in all the calculations made so far [1-3], although its meaning has not yet been fully discussed.

For the partial wave amplitudes at low Δ^2 the approximation used seems rather reasonable. In any case, we think it useful to investigate their Δ^2-dependence as carefully as possible, either as a way of getting some insight on the consistency of the approximations used at low Δ^2, or of approaching the problem of the high Δ^2 behaviour, where further difficulties arise at least from the presence of the form factors. This investigation will be carried out in the following sections.

3. – The off-shell amplitude.

The off-shell amplitude $M^{\alpha\beta}_{\pi N}(\omega^2, t^2, \Delta^2)$ defined in Sect. 2 can be written by invariance requirements as

$$M^{\alpha\beta}_{\pi N}(\omega^2, t^2, \Delta^2) = \bar{u}(q_1)[-A_{\alpha\beta} + i(\gamma\cdot q)B_{\alpha\beta}]n(p_1)\,, \tag{11}$$

where $A_{\alpha\beta}$ and $B_{\alpha\beta}$ depend on t^2, ω^2 and Δ^2 and can be written

$$A_{\alpha\beta} = \delta_{\alpha\beta}A^{(+)} + \tfrac{1}{2}[\tau_\beta, \tau_\alpha]A^{(-)}\,, \tag{12}$$

$$B_{\alpha\beta} = \delta_{\alpha\beta}B^{(+)} + \tfrac{1}{2}[\tau_\beta, \tau_\alpha]B^{(-)}\,, \tag{13}$$

$A^{(\pm)}$ are related to the off-shell amplitudes for fixed total isotopic spin through the relations

$$A^{(+)} = \tfrac{1}{3}(A^{(\frac{1}{2})} + 2A^{(\frac{3}{2})})\,, \qquad A^{(-)} = \tfrac{1}{3}(A^{(\frac{1}{2})} - A^{(\frac{3}{2})})\,, \tag{14}$$

and similarly for $B^{(\pm)}$. (12) and (13) imply an analogous splitting of $M_{\pi N}$ into a (+) and a (—) part.

By introducing two-dimensional spinors χ we get

$$M^{(\pm)}_{\pi N} = \frac{4\pi\omega}{m}\,\chi^+_f \left| f_1^{(\pm)} + \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{q}_1)(\boldsymbol{\sigma}\cdot\boldsymbol{p}_1)}{|\boldsymbol{q}_1||\boldsymbol{p}_1|} f_2^{(\pm)} \right| \chi_i\,. \tag{15}$$

Here $\boldsymbol{q}_1, \boldsymbol{p}_1$ are the 3-momenta of the nucleons calculated in the c.m. system of q_1 and q, $\sigma_i (i = 1, 2, 3)$ are the Pauli matrices, and

$$f_1^{(\pm)} = \frac{1}{4\pi} \frac{R_1}{2\omega} [A^{(\pm)} + (\omega - m) B^{(\pm)}] , \tag{16}$$

$$f_2^{(\pm)} = \frac{1}{4\pi} \frac{R_2}{2\omega} [- A^{(\pm)} + (\omega + m) B^{(\pm)}] , \tag{17}$$

with

$$\begin{cases} R_1 = [(p_{10} + m)(q_{10} + m)]^{\frac{1}{2}} \\ R_2 = [(p_{10} - m)(q_{10} - m)]^{\frac{1}{2}} . \end{cases} \tag{18}$$

Next we expand f_1 and f_2 in terms of off-shell scattering amplitudes with given parity and angular momentum. This expansion gives [6]

$$f_1^{(\pm)} = \sum_{l=0}^{\infty} f_{l_+}^{(\pm)}(\omega, \Delta^2) P'_{l+1}(\cos \varepsilon) - \sum_{l=2}^{\infty} f_{l_-}^{(\pm)}(\omega, \Delta^2) P'_{l-1}(\cos \varepsilon) , \tag{19}$$

$$f_2^{(\pm)} = \sum_{l=1}^{\infty} [f_{l_-}^{(\pm)}(\omega, \Delta^2) - f_{l_+}^{(\pm)}(\omega, \Delta^2)] P'_l(\cos \varepsilon) , \tag{20}$$

where ε is the scattering angle defined by (9).

It is obvious that all quantities defined by (11) to (20) reduce to the corresponding physical quantities defined in Ref. [6] as $\Delta^2 = -\mu^2$.

Let us consider now the « unitarity » condition, which can be derived from general principles when off-shell states are implied. For the case of matrix element (4) it can be shown [8] that the relation obtained is similar to the one valid on shell. In particular, for the partial amplitude $f_{l_\pm}(\omega, \Delta^2)$ one gets the following relation

$$f_{l\pm}^*(\omega, \Delta^2) = \exp [- 2i\delta_{l_\pm}(\omega)] f_{l_\pm}(\omega, \Delta^2) , \tag{21}$$

where $\delta_{l_\pm}(\omega)$ is the physical phase shift for $\pi.N$ scattering in the channel of angular momentum l and total angular momentum $l \pm \frac{1}{2}$. By introducing the

[8] The proof of formula (21) can be found in S. Fubini, Y. Nambu and V. Wataghin: *Phys. Rev.*, **111**, 329 (1957), Appendix 2, where invariance of the S matrix element under strong reflections is used. This procedure has been pointed out to us independently by Prof. V. Glaser.

physical transition amplitude $f_{l_\pm}(\omega) = (1/q)\exp[i\delta_{l_\pm}]\sin\delta_{l_\pm}(\omega)$, (21) transforms into

$$\text{Im}\, f_{l_\pm}(\omega, \Delta^2) = q f^*_{l_\pm}(\omega) f_{l_\pm}(\omega, \Delta^2)\,. \tag{22}$$

This relation will be used in the following developments.

4. – Dispersion relations.

We start with the assumption that the invariant amplitudes $A^{(\pm)}$ and $B^{(\pm)}$ obey dispersion relations of the form (9)

$$\left\{\begin{aligned} &\text{Re}\, A^{(\pm)}(\nu, t^2, \Delta^2) = \frac{\mathscr{P}}{\pi}\int_{\nu_{\min}}^{\infty} \text{Im}\, A^{(\pm)}(\nu', t^2, \Delta^2)\left(\frac{1}{\nu' - \nu} \pm \frac{1}{\nu' + \nu}\right) d\nu', \\ &\text{Re}\, B^{(\pm)}(\nu, t^2, \Delta^2) = \frac{G_\tau^2}{2m} K(\Delta^2)\left(\frac{1}{\nu_B - \nu} \mp \frac{1}{\nu_B + \nu}\right) + \\ &\qquad\qquad + \frac{\mathscr{P}}{\pi}\int_{\nu_{\min}}^{\infty} \text{Im}\, B^{(\pm)}(\nu', t^2, \Delta^2)\left(\frac{1}{\nu' - \nu} \mp \frac{1}{\nu' + \nu}\right) d\nu', \end{aligned}\right. \tag{23}$$

ν is the invariant

$$\nu = -\frac{1}{4m}(k + q)(p_1 + q_1)\,. \tag{24}$$

Furthermore

$$\nu_B = \frac{\Delta^2 - \mu^2}{4m} - \frac{t^2}{4m}\,, \tag{25}$$

$$\nu_{\min} = \mu + \frac{\Delta^2 + \mu^2}{4m} - \frac{t^2}{4m}\,. \tag{26}$$

The dispersion relations written in the form (23) take already into account the crossing invariance, which is maintained when the incident pion k goes off-shell.

(9) For the discussion of this point, see Sect. **7**.

For the following development it is convenient to change the variable of integration. Let us define

$$\nu_{\mathrm{L}} = \frac{\omega^2 - m^2 - \mu^2}{2m}, \tag{27}$$

$$b = \frac{\Delta^2 + \mu^2}{4m} - \frac{t^2}{4m}, \tag{28}$$

$$\nu_0 = -\frac{\mu^2}{2m}. \tag{29}$$

We have then

$$\begin{cases} \nu \;\; = \nu_{\mathrm{L}} + b, \\ \nu_B \;\; = \nu_0 + b, \\ \nu_{\min} = \mu + b, \end{cases} \tag{30}$$

ν_{L} represents the total lab. energy of an incoming real π meson at total c.m. energy ω. The dispersion relations (23), rewritten in terms of ν_{L}, read

$$\begin{cases} \mathrm{Re}\, A^{(\pm)}(\nu_{\mathrm{L}}, t^2, \Delta^2) = \dfrac{\mathscr{P}}{\pi}\displaystyle\int_{\mu}^{\infty} \mathrm{d}\nu_{\mathrm{L}}'\, \mathrm{Im}\, A^{(+)}(\nu_{\mathrm{L}}', t^2, \Delta^2)\left(\frac{1}{\nu_{\mathrm{L}}' - \nu_{\mathrm{L}}} \pm \frac{1}{\nu_{\mathrm{L}}' + \nu_{\mathrm{L}} + 2b}\right), \\ \mathrm{Re}\, B^{(\pm)}(\nu_{\mathrm{L}}, t^2, \Delta^2) = \dfrac{G_r^2}{2m} K(\Delta^2)\left(\frac{1}{\nu_0 - \nu_{\mathrm{L}}} \mp \frac{1}{\nu_0 + \nu_{\mathrm{L}} + 2b}\right) + \\ \qquad + \dfrac{\mathscr{P}}{\pi}\displaystyle\int_{\mu}^{\infty} \mathrm{d}\nu_{\mathrm{L}}'\, \mathrm{Im}\, B^{(\pm)}(\nu_{\mathrm{L}}', t^2, \Delta^2)\left(\frac{1}{\nu_{\mathrm{L}}' - \nu_{\mathrm{L}}} \mp \frac{1}{\nu_{\mathrm{L}}' + \nu_{\mathrm{L}} + 2b}\right). \end{cases} \tag{31}$$

By following the procedure of *CGLN*, we take under the integrals only the 3.3 resonant wave, which gives the main contribution to the imaginary part of $A^{(\pm)}$ and $B^{(\pm)}$. We can then take, if ν_{L} is not too large, all the partial waves up to d-waves, and get from (19)

$$f_1^{(\pm)} \simeq f_S^{(\pm)} - f_{d\frac{3}{2}}^{(\pm)} + 3\cos\varepsilon\, f_{p\frac{3}{2}}^{(\pm)} + \tfrac{3}{2}(5\cos^2\varepsilon - 1) f_{d\frac{3}{2}}^{(\pm)}. \tag{32}$$

and by deriving with respet to t^2, and using (9), we get

$$-\frac{3}{2q_1 p_1} f_{p\frac{3}{2}}^{(\pm)} \simeq \left|\frac{\partial}{\partial t^2} f_1^{(\pm)}\right|_{t^2=0} + 2(q_{10}p_{10} - m^2)\left|\frac{\partial^2}{\partial (t^2)^2} f_1^{(\pm)}\right|_{t^2=0}. \tag{33}$$

We can then obtain from (31) a relation containing $f^{(\pm)}_{p\frac{3}{2}}$ alone by the following procedure. Take the linear combination (16) of eqs. (31) which reproduces the real part of $f_1^{(\pm)}$ as a function of the imaginary parts of $A^{(\pm)}$ and $B^{(\pm)}$. Derive with respect to t^2 and set $t^2 = 0$. From the linear combination (34) which will reproduce the real part of $f^{(\pm)}_{p\frac{3}{2}}$. Express the quantities Im A and Im B under the integrals as a function of the resonant phase alone. The following relations, obtainable by inverting (16) and (17), have to be used:

$$\text{Im}\, A^{(\pm)} = 4\pi \left| \frac{\omega + m}{R_1} \text{Im}\, f_1^{(\pm)} - \frac{\omega - m}{R_2} \text{Im}\, f_2^{(\pm)} \right| , \tag{34}$$

$$\text{Im}\, B^{(\pm)} = 4\pi \left| \frac{1}{R_1} \text{Im}\, f_1^{(\pm)} + \frac{1}{R_2} \text{Im}\, f_2^{(\pm)} \right| , \tag{35}$$

where

$$\begin{cases} \text{Im}\, f_1^{(\pm)} \simeq 3 \cos\varepsilon \, \text{Im}\, f^{(\pm)}_{p\frac{3}{2}} , \\ \text{Im}\, f_2^{(\pm)} \simeq - \text{Im}\, f^{(\mp)}_{p\frac{3}{2}} , \end{cases} \tag{36}$$

under the dispersive integrals. The final result of this manipulation is

$$\begin{aligned} \text{Re}\, f^{(\pm)}_{p\frac{3}{2}} = - \frac{p_1 q_1}{3\omega} R_1 \Bigg\{ \mp \frac{G_\tau^2}{4\pi} \frac{\omega - m}{4m^2} \frac{K(\Delta^2)}{(\nu_0 + \nu_L + \beta/2m)^2} \cdot \\ \cdot \left[1 + \frac{(2/m)(p_{10} q_{10} - m^2)}{\nu_0 + \nu_L + \beta/2m} \right] + \frac{\mathscr{P}}{\pi} \int_\mu^\infty d\nu'_L \frac{\text{Im}\, f^{(\pm)}_{p\frac{3}{2}}(\nu'_L, \Delta^2)}{p'_1 q'_1} \cdot \\ \cdot \Bigg\{ - \frac{3}{2} \frac{\omega' + \omega}{R'_1} \cdot \frac{1}{\nu'_L - \nu_L} \pm \left(- \frac{3}{2} \right) \frac{\omega' - \omega + 2m}{R'_1 (\nu'_L + \nu_L + \beta/2m)} \left| 1 + \frac{(2/m)(p_{10} q_{10} - m^2)}{\nu'_L + \nu_L + \beta/2m} \right| \pm \\ \pm \frac{1}{2m} \left| \frac{\omega' - \omega + 2m}{R'_1} \cdot 3(p'_{10} q'_{10} - m^2) + R'_1 (\omega' + \omega - 2m) \right| \cdot \\ \cdot \left[1 + \frac{(2/m)(p_{10} q_{10} - m^2)}{\nu'_L + \nu_L + \beta/2m} \right] \cdot \frac{1}{(\nu'_L + \nu_L + \beta/2m)^2} \Bigg\} \Bigg\} . \end{aligned} \tag{37}$$

We have put $\beta = \Delta^2 + \mu^2$, and made use of the identity $R'_1 R'_2 = p'_1 q'_1$. This equation, in the limit $\beta \to 0$ (which implies $p_{10}, p_1 \to q_{10}, q_1$; $R_1 \to q_{10} + m$) reduces to eq. (3.32) of CGLN.

Since from (14) it can be deduced that

$$\text{Re}\, f_{33} = \text{Re}\, f^{(+)}_{p\frac{3}{2}} - \text{Re}\, f^{(-)}_{p\frac{3}{2}} , \tag{38}$$

we form the combination (38) of eqs. (37), use the relations

$$\text{(39)} \qquad \begin{cases} \operatorname{Im} f^{(+)}_{p\frac{3}{2}} \simeq \frac{2}{3} \operatorname{Im} f_{33} , \\ \operatorname{Im} f^{(-)}_{p\frac{3}{2}} \simeq -\frac{1}{3} \operatorname{Im} f_{33} , \end{cases}$$

and finally get

$$\text{(40)} \quad \operatorname{Re} g_{33}(\omega, \Delta^2) = B + \frac{\mathscr{P}}{\pi} \int_{\mu}^{\infty} \mathrm{d}\nu'_{\mathrm{L}} \operatorname{Im} g_{33}(\omega', \Delta^2) \cdot \cdot \left[\frac{\omega' + \omega}{2\omega'} \cdot \frac{1}{\nu'_{\mathrm{L}} - \nu_{\mathrm{L}}} + \frac{1}{3} \varrho(\omega, \omega', \Delta^2) \frac{1}{\nu'_{\mathrm{L}} + \nu_{\mathrm{L}} + \beta/2m} \right],$$

where

$$\text{(41)} \qquad g_{33}(\omega, \Delta^2) = \frac{1}{p_1 q_1} \frac{2\omega}{R_1} f_{33}(\omega, \Delta^2) ,$$

$$\text{(42)} \qquad B(\omega, \Delta^2) = \frac{4}{3} \frac{f^2}{\mu^2} \frac{(\omega - m) K(\Delta^2)}{(\nu_0 + \nu_{\mathrm{L}} + \beta/2m)^2} \left[1 + \frac{(2/m)(q_{10} p_{10} - m^2)}{\nu_0 + \nu_{\mathrm{L}} + \beta/2m} \right],$$

$$\text{(43)} \quad \varrho(\omega, \omega', \Delta^2) = \left\{ \frac{\omega' - \omega + 2m}{2\omega'} - \frac{1/m}{\nu'_{\mathrm{L}} + \nu_{\mathrm{L}} + \beta/2m} \cdot \right.$$
$$\left. \cdot \left[\frac{\omega' - \omega + 2m}{2\omega'} \cdot (q'_{10} p'_{10} - m^2) + \frac{\omega' + \omega - 2m}{6\omega'} R'^2_1 \right] \right\} \cdot \left[1 + \frac{(2/m)(q_{10} p_{10} - m^2)}{\nu'_{\mathrm{L}} + \nu_{\mathrm{L}} + \beta/2m} \right].$$

5. – The integral equation.

We will show in this section how eq. (40) can be transformed into an integral equation for g_{33}. From (22) and (41) it follows

$$\text{(44)} \qquad \operatorname{Im} g_{33}(\omega, \Delta^2) = \exp[-i\delta_{33}] \sin \delta_{33} g_{33}(\omega, \Delta^2) ,$$

where δ_{33} is the physical $T = J = \frac{3}{2}$ πN phase shift. Eq. (40) can thus be written

$$\text{(45)} \quad g_{33}(\omega, \Delta^2) = B + \frac{1}{\pi} \int_{\mu}^{\infty} \mathrm{d}\nu'_{\mathrm{L}} \exp[-i\delta'_{33}] \sin \delta'_{33} g_{33}(\omega', \Delta^2) \cdot \cdot \left[\frac{1}{\nu'_{\mathrm{L}} - \nu_{\mathrm{L}} - i\varepsilon} \frac{\omega' + \omega}{2\omega'} + \frac{\varrho}{3} \frac{1}{\nu'_{\mathrm{L}} + \nu_{\mathrm{L}} + \beta/2m} \right].$$

This equation is of the general type discussed by OMNÈS (7) (one can also take the factor $(\omega'+\omega)/2\omega'$ into account, by slightly modifying Omnès' procedure). Therefore (45) can be solved exactly. The condition $\delta_{33}(v_L = \mu) = 0$ ensures the uniqueness of the solution. We can then obtain in this way the expression of the off-shell low energy 3.3 amplitude for arbitrary Δ^2. We will write down its expression only in the static limit.

Let us then go to the static limit by leaving Δ^2 arbitrary. If we put

$$u = \frac{\omega - m}{m}, \tag{46}$$

and consider values of u small with respect to unity, we can prove that

$$|\varrho| \leqslant \tfrac{1}{3}, \tag{47}$$

for any $\Delta^2 \geqslant -\mu^2$ and up to $u, u' \sim 0.45$.

The second term in the integral of (40) or (45) can therefore be neglected, as CGLN did. We note that in our case this approximation is even better due to the presence of the term $\beta/2m$ in the denominator. Furthermore

$$\frac{\omega'+\omega}{2\omega'} \simeq 1; \qquad \frac{R_1}{2\omega} \simeq \left(1+\frac{\beta}{4m^2}\right)^{\frac{1}{2}}, \tag{48}$$

$$\frac{v_L}{m} \simeq u, \tag{49}$$

and

$$B \simeq \frac{4f^2 u}{3m\mu^2} \frac{K(\Delta^2)}{(u+\beta/2m^2)^2}\left[1+\frac{\beta/m^2}{u+\beta/2m^2}\right]. \tag{50}$$

With these approximations (45) simplifies greatly and reduces to

$$g_{33}(u, \Delta^2) = B + \frac{1}{\pi}\int_{\mu/m}^{\infty} \mathrm{d}u' \, \frac{\exp[-i\delta'_{33}]\sin\delta'_{33}}{u'-u-i\varepsilon} g_{33}(u', \Delta^2), \tag{51}$$

whose solution can be obtained from formula (2.11) of ref (7). In the next section we will discuss the solution of this equation more in detail.

6. – The solutions of the integral equation.

In this section we discuss the solution of our integral equation in the approximate form (51). As it has been shown in Ref. [7], the expression of the correct solution is

$$(52) \quad g_{33}(u, \Delta^2) = \exp[i\delta_{33}]\left\{ B(u, \Delta^2)\cos\delta_{33} + \Omega(u)\frac{\mathscr{P}}{\pi}\int_{\mu/m}^{\infty}\frac{\sin\delta'_{33}B(u', \Delta^2)}{(u'-u)\Omega(u')}\,\mathrm{d}u' \right\},$$

where

$$(53) \quad \Omega(u) = \exp\left\{ \frac{\mathscr{P}}{\pi}\int_{\mu\, m}^{\infty}\frac{\delta_{33}(x)}{x-u}\,\mathrm{d}x \right\}.$$

In practice, however, eq. (52) does not seem to be very suitable for numerical calculations. The principal difficulty lies in the fact that the function $\Omega(u)$ which enters under the integral sign is in turn a complicated integral which requires, alone, machine computations. It would then be preferable to obtain an approximate expression for (52) which does not contain integrations to be performed numerically.

Let us first rewrite (52) in the following form

$$(54) \quad K^{-1}(\Delta^2)g_{33}(u, \Delta^2)\exp[-i\delta_{33}] = B_0(u, \Delta^2)\cos\delta_{33} + \\ + \Omega(u)\frac{\mathscr{P}}{\pi}\int_{\mu/m}^{\infty}\frac{\sin\delta'_{33}B_0(u', \Delta^2)}{(u'-u)\Omega(u')}\,\mathrm{d}u',$$

where

$$(55) \quad B_0(u, \Delta^2) = K^{-1}(\Delta^2)B(u, \Delta^2).$$

Consider first the case of Δ^2 small ($\Delta^2 \leqslant 5\mu^2$). It can easily be checked by developing (50) in powers of $\beta/2m^2u \ll 1$, that

$$(56) \quad B_0(u, \Delta^2) \simeq \frac{4f^2}{3m\mu^2u} \equiv B(u, -\mu^2),$$

since the corrections to eq. (56) are of the second order in $\beta/2m^2u$ [10]. (56) still represents the correct Born term within a 10% approximation up to $\Delta^2 \sim 5\mu^2$.

By inserting (56) in (54) one gets

$$K^{-1}(\Delta^2)g_{33}(u,\Delta^2)\exp[-i\delta_{33}] \simeq B(u,-\mu^2)\cos\delta_{33} + \\ + \Omega(u)\frac{\mathscr{P}}{\pi}\int_{\mu/m}^{\infty}\frac{\sin\delta'_{33}B(u',-\mu^2)}{(u'-u)\Omega(u')}\,\mathrm{d}u'. \tag{57}$$

In this equation, the r.h.s. does not depend on Δ^2. Therefore, it can be identified by going to the limit $\Delta^2 \to -\mu^2$, where the l.h.s. becomes simply $\sin\delta_{33}\, q_1^{-3}$. From (57), it can then be deduced

$$g_{33}(u,\Delta^2) \simeq K(\Delta^2)\frac{\exp[i\delta_{33}]\sin\delta_{33}}{q_1^3}, \tag{58}$$

or

$$f_{33}(u,\Delta^2) \simeq K(\Delta^2)\frac{p_1}{q_1}\frac{\exp[i\delta_{33}]\sin\delta_{33}}{q_1}. \tag{59}$$

Eq. (59) gives us the correct extrapolation of the resonant amplitude for small Δ^2. We see that the off-shell amplitude differs from the on-shell amplitude only by a Δ^2-dependent proportionality factor.

Now let us consider an arbitrary positive value of Δ^2. Eq. (54) can be written

$$K^{-1}(\Delta^2)g_{33}(u,\Delta^2)\exp[-i\delta_{33}] = B_0(u,\Delta^2)\cos\delta_{33} + \\ + \Omega(u)\frac{\mathscr{P}}{\pi}\int_{\mu/m}^{\infty}\frac{\sin\delta'_{33}B(u',-\mu^2)}{(u'-u)\Omega(u')}\gamma(u',\Delta^2)\,\mathrm{d}u', \tag{60}$$

where

$$\gamma(u,\Delta^2) = \frac{B_0(u,\Delta^2)}{B(u,-\mu^2)} = \frac{1+3\alpha}{(1+\alpha)^3}, \tag{61}$$

with

$$\alpha = \beta/2m^2u. \tag{62}$$

(10) It is interesting to note that this cancellation occurs also in the neglected ϱ term. The only terms linear in β which survive in it are of the order $\beta/2m^2$ and can be neglected in the present approximation.

It can be easily seen that, for fixed Δ^2, $\gamma(u, \Delta^2)$ is a positive function of u which increases from a minimum value to 1 as u ranges from threshold to infinity. Therefore, the convergence of the integral in the r.h.s. of (60) is not changed with respect to the on-shell case (eq. (57)). The function $\gamma(u, \Delta^2)$ is a sufficiently smooth function of u as to leave the integral appearing in the r.h.s. of (60) still dominated by values of u' near the resonance. We put then

$$(63)\quad \Omega(u)\frac{\mathscr{P}}{\pi}\int_{\mu/m}^{\infty}\frac{\sin\delta'_{33}\,B(u',-\mu^2)}{(u'-u)\,\Omega(u')}\,\gamma(u',\Delta^2)\,\mathrm{d}u' \simeq$$

$$\simeq \gamma(u_r,\Delta^2)\,\Omega(u)\,\frac{\mathscr{P}}{\pi}\int_{\mu/m}^{\infty}\frac{\sin\delta'_{33}\,B(u',-\mu^2)}{(u'-u)\,\Omega(u')}\,\mathrm{d}u'.$$

Eq. (63) becomes less accurate with increasing Δ^2, but the errors arising from the approximation made in it are estimated not to exceed 15% even in the most pessimistic case.

Since from the limit $\Delta^2 = -\mu^2$ of eq. (57) it can be deduced

$$(64)\qquad \Omega(u)\,\frac{\mathscr{P}}{\pi}\int_{\mu/m}^{\infty}\frac{\sin\delta'_{33}\,B(u',-\mu^2)}{(u'-u)\,\Omega(u')}\,\mathrm{d}u' = \frac{\sin\delta_{33}}{q_1^3} - B(u,-\mu^2)\cos\delta_{33}\,,$$

eq. (60) can be transformed in

$$(65)\quad K^{-1}(\Delta^2)\,g_{33}(u,\Delta^2)\exp[-i\delta_{33}] \simeq B_0(u,\Delta^2)\cos\delta_{33} +$$

$$+\,\gamma(u_r,\Delta^2)\left[\frac{\sin\delta_{33}}{q_1^3} - B(u,-\mu^2)\cos\delta_{33}\right],$$

from which one obtains (cf. (41), (48))

$$(66)\quad f_{33}(u,\Delta^2) \simeq K(\Delta^2)\left(1+\frac{\beta}{4m^2}\right)^{\frac{1}{2}}\frac{p_1}{q_1}\exp[i\delta_{33}]\,\gamma(u_r,\Delta^2)\cdot$$

$$\cdot\left\{\frac{\sin\delta_{33}}{q_1} + q_1^2\left[\frac{B(u,\Delta^2)}{\gamma(u_r,\Delta^2)} - B(u,-\mu^2)\right]\cos\delta_{33}\right\},$$

which gives the desired extrapolation for arbitrary Δ^2. We note now that the term containing $\cos\delta_{33}$ in (66) vanishes at the resonance, because both $\cos\delta_{33}$ and its multiplying factor are zero: furthermore, this term is always rather small and represents a correction to the dominant term containing $\sin\delta_{33}$, which is then proportional to the physical amplitude.

7. – Concluding remarks.

Under certain assumptions and some approximations, we have solved the problem of obtaining the M-matrix element for « off-shell » πN scattering in terms of the « on-shell » matrix element. The integral eq. (45) is a rigorous consequence of the following two hypotheses: the validity of the dispersion relations (23) and the dominance of the off-shell $T = J = \frac{3}{2}$ amplitude under the dispersive integrals.

The validity of the first assumption can be proved by modifying the well-known methods of deriving the usual dispersion relations. In the proof of the usual dispersion relations both pions are considered off-shell and analytic continuation is made from unphysical values of the pion « mass » ($\Delta^2 > 0$, as in our case) to the physical value $-\mu^2$. However, the masses of the two pions are kept equal in this procedure. In our case one should continue only one of the two masses and keep the other in the unphysical region. This procedure has been shown to be possible by Bros, Froissart, Omnès and Stora [11].

The second assumption is difficult to justify from a purely theoretical standpoint. Consistency arguments [6] have indicated that in the on-shell case it is probably a good assumption [12]. In the off-shell case it should still be rather good unless dramatic changes are induced by the Δ^2-dependence. The experimental indications, as discussed in the introduction, are so far against such a strong dependence.

As it has already been stated, eq. (45) can be solved exactly with standard methods. A numerical evaluation of the solution requires, however, complicated machine calculations, and we thought it useful to obtain an approximate explicit formula for the off-shell amplitude, given by eq. (66). The approximations made in order to obtain this expression can be summarized as follows. First, the rigorous eq. (45) has been reduced to the eq. (51) by going to the static limit [13]. The solution of such an equation has been worked out by comparison with the limit obtained from (51) itself in the on-shell case. By this procedure, in practice, we have assumed that, if one takes the limit $\Delta^2 = -\mu^2$ in eq. (52) and introduces a phenomenological expression for quantities containing δ_{33}, the equation obtained should become an identity (and this implies a strict

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[11] R. Omnès: « *Les Houches* », *Summer Course on Dispersion Relations*, (Paris, 1960).

[12] The main criticism to the work by CGLN, which concerns the neglect of the $T=J=1$ $\pi\pi$ resonance, does not regard the $T=J=\frac{3}{2}$ equation, which is very little influenced by this system. See J. Bowcock, N. W. Cottingham and D. Lurié: *Nuovo Cimento*, **16**, 918 (1960).

validity of the assumption made in Ref. (6) that in the on-shell case the dispersive integrals are dominated by the 3.3 resonance). However, we think that this procedure has to be considered as an approximate one, since this « identity » might not be verified rigorously. The last approximation introduced consists in taking the smoothly varying function $\gamma(u', \Delta^2)$ outside the integral sign. We think that the over-all uncertainty in the solution induced by these approximations should not exceed $\sim 20\%$ (14).

The result obtained for $M_{\pi\mathcal{N}}(\omega, t^2, \Delta^2)$ can allow us to calculate correctly the contribution of a peripheral graph to an inelastic process, by inserting (66) into (19) and (20), when the other non-resonant off-shell partial wave amplitudes for $\pi\mathcal{N}$ scattering can be neglected. A feature which should be remarked is the occurrence of $K(\Delta^2)$ as a multiplicative factor in the solution obtained for $f_{33}(u, \Delta^2)$ (expressions (59) and (66)). When the analytic continuation of $M_{\pi\mathcal{N}}$ (ω, t^2, Δ^2) is put into (4), we see that the only unknown function of Δ^2 entering in this formula is the product $K^2(\Delta^2)K'(\Delta^2)$. It is not completely out of question as to whether an experimental estimate of this quantity can be made: in fact the many-pion exchanges in process (3) probably give contributions which depend on the energy in a different way than the one-pion diagram. If the predictions of the one-pion exchange theory (calculated with $K^2K' = 1$) diverged from the experimental data at various energies only by a Δ^2 (and not energy) dependence, one could reasonably attribute the error to the effect of the neglected form factors, and try to determine K^2K' from experiment.

* * *

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(13) To go to the static limit means, as usually, to consider $u = (\omega - m)/m$ small with respect to unity: this approximation can, of course, be performed also when an off-shell π-meson is present.

(14) In any case all the approximations performed can be tested by comparison of (63) with the solution of the rigorous integral eq. (45).

Note added in proof.

Considerations very similar to those contained in the present paper have been made by J. Iizuka and A. Klein: *Progr. Theor. Phys.*, **25**, 1017 (1961). This article came to our attention after completion of the present work. The solution obtained by the above authors is limited to the region of small Δ^2 ($\Delta^2 \simeq \mu^2$) and is there practically coincident with our solution.

RIASSUNTO

Si discute il problema dello scattering pione-nucleone quando uno dei pioni è virtuale. Assumendo che si possano scrivere relazioni di dispersione unidimensionali nella variabile energia per ampiezze invarianti opportunamente definite, e che anche nel caso « off shell » gli integrali dispersivi siano dominati dall'ampiezza $T=J-\frac{3}{2}$, si deduce per tale ampiezza un'equazione integrale del tipo discusso da Omnès. La soluzione esatta di tale equazione è discussa e ricondotta ad una forma approssimata più semplice e più conveniente per calcoli numerici. Tale forma è particolarmente semplice per valori del quadrato del quadri-impulso del pione virtuale minori di ~ 5 masse pioniche quadrate.

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Unscharfes Raum-Zeit-Kontinuum, unscharfe Masse und Pauli-Villars-Regularisierung.

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(ricevuto il 10 Luglio 1961)

Summary. — It is shown, that the field equations for a non-localizable, scalar field of mass zero (deduced in [3]) have (among many others) solutions, the Green's functions of which are identical with the regularized functions of Pauli and Villars [11]. The compensating masses however have here a physical meaning, insofar, as they correspond to the different possible masses of one particle with uncertain mass. The result of this work is then that it should be possible to create a convergent field-theory by introducing into the theory an uncertain (inside a very small interval, *e.g.*) mass of the field-particle, or (what is the same in this special example) uncertain positions of the test-bodies, measuring the field.

1. – Einführung.

Unsere Überlegungen [1] bis [8] zur Frage der Meßbarkeit einfacher Felder führten uns darauf, daß in der Quantentheorie dieser Felder neben den Feldoperatoren $U(y)$ noch andere Operatoren $\xi_\mu(y)$ auftreten sollten, die die 4-dimensionale Stelle repräsentieren, an der die den Feldwert $U(y)$ liefernde Messung

[1] G. Heber: *Nuovo Cimento*, **7**, 677 (1958).
[2] G. Heber: *Nuovo Cimento*, **8**, 327 (1958).
[3] G. Heber: *Nuovo Cimento*, **12**, 553 (1959).
[4] G. Heber: *Max Planck-Festschrift* (1958), S. 345.
[5] G. Heber: *Wiss. Zs. Univ. Jena Jahrg.*, **8**, 357 (1959).
[6] G. Heber: *Wiss. Zs. Univ. Jena Jahrg.*, **9**, 455 (1960).
[7] G. Heber: *Acta Phys. Pol.*, **18**, 581 (1959).
[8] G. Heber: *Acta Phys. Hung.*, **12**, 297 (1961).

ausgeführt wurde. Der folgerichtige Einbau dieser Größen $\xi_\mu(y)$ in den Formalismus des klassischen Feldes

$$\Box U = 0 \tag{1}$$

führte zu 2 wesentlich neuen Resultaten:

1). Die Feldgleichung läßt sich jetzt in der Form

$$\partial_\mu[(\partial_\nu \Phi) K^{\mu\nu}] = 0 \tag{2a}$$

oder

$$(\partial_\mu \partial_\nu \Phi) K^{\mu\nu} + (\partial_\nu \Phi)\, \partial_\mu K^{\nu\mu} = 0 \tag{2b}$$

schreiben, wo sich die Operatoren $K^{\nu\mu}(y)$ von den $\xi^\mu(y)$ ableiten, während $\Phi(y)$ als neue Feldgröße aus U und ξ_μ zusammengesetzt wird.

2). Wenn man annimmt, daß die Kommutatoren zwischen den U c-Zahlen sind (wie in der konventionellen Theorie isolierter Felder), so werden die Kommutatoren zwischen den Φ q-Zahlen. Man kann diese q-Zahl-Kommutatoren aber noch nicht vollständig angeben, weil die Eigenschaften der $K_{\mu\nu}$ bzw. ξ_μ durch unsere bisherigen Modellbetrachtungen nicht ausreichend fixiert sind.

Autor hat die Hoffnung, konnte aber bisher nicht beweisen, daß die modifizierte, den Meßmöglichkeiten angepaßte Theorie keine Divergenzschwierigkeiten besitzt. Zunächst wurde untersucht [6], ob man etwa unter Beibehaltung von (1), nur durch Ersatz von c-Zahl-Kommutatoren durch irgendwelche mit (1) verträgliche q-Zahl-Kommutatoren an den Divergenzschwierigkeiten der Quantenfeldtheorie irgendetwas ändern kann. Es konnte gezeigt werden, daß man nichts an diesen Schwierigkeiten ändern kann, solange man verlangt, daß die zu bauende Theorie mikroskopisch kausal im y-Raum ist.

Hier stehen nun die Fragen zur Debatte, welcher Art die Lösungen von (2) sind bzw. ob der Übergang von (1) zu (2) die Möglichkeit einer Beseitigung der Divergenzen enthält. Diese Fragen können natürlich nur beantwortet werden, wenn man über die Eigenschaften der $K^{\mu\nu}$-Operatoren genügend weiß. Trotz längerer Überlegungen ist es dem Autor nicht gelungen, aus Modellbetrachtungen die erforderlichen Informationen über $K^{\mu\nu}(y)$ zu erhalten. Verfasser neigt deshalb z.Zt. zu der Überzeugung, daß Modellbetrachtungen überhaupt prinzipiell nicht ausreichen, eine Quantentheorie festzulegen. Deshalb wenden wir hier die Frage so: Gibt es *überhaupt* Operatoren $K_{\mu\nu}(y)$, die in (2) eingesetzt dazu führen, daß die dabei entstehende Quantenfeldtheorie günstigere Eigenschaften besitzt als die konventionelle? Ob die in dieser Arbeit zugelassenen $K_{\mu\nu}(y)$ sich dann irgendwie in vernünftiger Weise mit $\xi_\mu(y)$-Operatoren darstellen lassen, soll in einer späteren Publikation untersucht werden.

Natürlich kann man die so gestellte Frage nicht in voller Allgemeinheit beantworten. Wir werden in der Regel solche $K_{\mu\nu}(y)$ ins Auge fassen, die (abgesehen von der trivialen Lösung $K_{\mu\nu}(y) = \delta_{\mu\nu}$ (*)) am einfachsten sind.

Im Anhang wird diskutiert, ob es etwa vernünftig wäre zu versuchen, Gleichungen für die $K_{\mu\nu}$ aus einem Variationsprinzip zu ermitteln. Dabei erhalten wir ein negatives Resultat.

2. – Eine erste Umformung der Feldgleichung (2).

Wir beginnen mit einigen wesentlichen Voraussetzungen über die Operatoren $K_{\mu\nu}(y)$. Die gesamte Theorie soll ja invariant im y-Raum gegen inhomogene Lorentztransformationen sein. Da sich die Feldgrößen einer solchen Theorie bei Translationen wie

$$\Phi(y) = U(y)\,\Phi(0)\,U^{-1}(y)\;; \tag{3}$$

$$U(y) = \exp\left[i\widetilde{P}_\lambda y^\lambda\right] \quad (**) \tag{4}$$

transformieren, müssen die K sich mit demselben U transformieren:

$$K_{\mu\nu}(y) = U(y)\,K_{\mu\nu}(0)\,U^{-1}(y)\,. \tag{5}$$

Für die Operatoren $K_{\mu\nu}(0)$ führen wir die durchaus völlig willkürliche Annahme

$$K_{\mu\nu}(0) = \delta_{\mu\nu}\cdot\Gamma \tag{6}$$

ein. $\Gamma = 1$ würde auf die konventionelle Theorie zurückführen. Wir wollen aber Γ als q-Zahl $\neq 1$ ansehen. Die Festlegung (6) wird hier nur durch die Kriterien der Einfachheit und der Korrespondenz zur konventionellen Theorie nahegelegt. — Aus (2*b*) wird mit (5) und (6):

$$\left(\Box\,\Phi(y)\right) U(y)\Gamma U^{-1}(y) + \left(\partial_\nu \Phi(y)\right)\partial^\nu[U(y)\Gamma U^{-1}(y)] = 0\,. \tag{7}$$

(*) $$\delta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix}.$$

(**) $\widetilde{P}_\lambda$ ist vermutlich *nicht* identisch mit dem Vierervektor von Energie und Impuls der Theorie; siehe auch [8], Abschnitt 11.

Wegen der Lorentzinvarianz sollte für jeden der Theorie angehörenden Operator $F(y)$ die Relation

$$[\tilde{P}^\nu, F(y)] = i\,\partial^\nu F(y) \tag{8}$$

gelten ([8]). Also können wir schreiben:

$$\partial^\nu[U(y)\Gamma U^{-1}(y)] = -\,i[\tilde{P}^\nu, U(y)\Gamma U^{-1}(y)] = -\,i\,U(y)[\tilde{P}^\nu, \Gamma]\,U^{-1}(y)\,.$$

(Die hier auftretende Größe $[\tilde{P}^\nu, \Gamma]$ kann im Rahmen unserer Absichten offenbar noch weitgehend willkürlich gewählt werden, denn wir kennen ja die explizite Gestalt von $\tilde{P}^\nu$ noch nicht.)

Jetzt multiplizieren wir (7) von rechts mit $(U(y)\Gamma U^{-1}(y))^{-1}$ und erhalten:

$$\Box\,\Phi(y) + \big(\partial_\nu \Phi(y)\big)\,b^\nu(y) = 0\;; \tag{9}$$

mit

$$b^\nu(y) = -\,i\,U(y)[\tilde{P}^\nu, \Gamma]\cdot\Gamma^{-1}\cdot U^{-1}(y)\,. \tag{10}$$

[Man kann auch schreiben:

$$\left|\begin{array}{l} b^\nu(y) = -\,i\,U(y)(\tilde{P}^\nu - \Gamma\tilde{P}^\nu\Gamma^{-1})\,U^{-1}(y)\,, \\ \text{oder} \\ b^\nu(y) = U(y)\,b^\nu(0)\,U^{-1}(y)\,, \\ \text{mit} \\ b^\nu(0) = -\,i(\tilde{P}^\nu - \Gamma\tilde{P}^\nu\Gamma^{-1}).] \end{array}\right. \tag{11}$$

3. – Über die Abhängigkeit der b^ν von y.

Bei linearen Feldgleichungen kann man Lösungen oft durch formale Fourier-Transformation erhalten. Wir wollen deshalb (9) einer solchen Transformation unterwerfen, wobei sich zeigen wird, daß (9) physikalisch vernüftige Lösungen möglicherweise dann enthält, wenn die b^ν unabhängig von y sind.

Setzer wir also an:

$$\Phi(y) = (2\pi)^{-2}\int a_k \exp[iky]\,\mathrm{d}^4k\,; \qquad b^\nu(y) = (2\pi)^{-2}\int b^\nu \exp[iky]\,\mathrm{d}^4k\,.$$

Dann ergibt sich aus (9):

$$a_k \cdot k_\nu k^\nu = c_k = (2\pi)^{-2} \int a_x \cdot i x_\nu b^\nu_{x-k} \, \mathrm{d}^4 x \,. \tag{12}$$

[In der konventionellen Theorie steht an dieser Stelle die Gleichung $a_k \cdot k^\nu k_\nu = 0$. Daraus folgt: entweder $a_k = 0$ oder $k_\nu k^\nu = 0$. D.h. die durch $k^\nu k_\nu = 0$ gekennzeichneten kinematischen Eigenschaften sind von dem durch a_k gekennzeichneten Besetzungsgrad des betreffenden Zustandes unabhängig, wie das bei einem isolierten Felde sein sollte.]

In (12) hingegen sind die a_k mit den k_ν i.a. unlösbar verknüpft. Es hängt z.B. das Verhalten eines Teilchens in einem bestimmten k-Zustand davon ab, welche anderen Zustände wie stark besetzt sind (wegen des Integrals rechts). Das ist u.E. ein für ein isoliertes Feld sehr schwer verständliches Verhalten, deshalb möchten wir daraus schließen, daß b^ν nicht von y abhängen sollte. Dann wird nämlich aus (9) mit

$$\Phi(y) = (2\pi)^{-2} \int a_k \exp\left[iky\right] \mathrm{d}^4 k \,,$$

einfach $a_k(k_\nu k^\nu - i k_\nu b^\nu) = 0$ d.h.: entweder $a_k = 0$ oder

$$k_\nu k^\nu - i k_\nu b^\nu = 0 \,. \tag{13}$$

Hier ist die Kinematik unabhängig vom Besetzungsgrad, wie üblich. — Die Relation

$$b^\nu(y) = b^\nu(0) \tag{14}$$

folgt z.B. aus (11), wenn man annimmt, daß $b_\nu(0)$ mit $\tilde{P}^\lambda$ kommutiert. Dies sei ab jetzt ausdrücklich vorausgesetzt.

Wir bemerken, daß der Schluß auf (14) natürlich nicht zwingend ist. Man kann aber sagen, daß durch Verwendung von (14) die Theorie viel einfacher wird, weshalb wir uns hier zuerst mit dieser Version der Theorie befassen wollen.

Wären die b^ν c-Zahlen, hätten wir in (13) schon einen wesentlichen Bestandteil der Lösung vor uns. Die konventionelle Relation $k^\nu k_\nu = 0$ würde einfach durch (13) zu ersetzen sein. D.h. für kleine k_ν gilt noch genähert die konventionelle Beziehung zwischen den k_ν (Bewegung mit Lichtgeschwindigkeit), für große k aber gibt es starke Abweichungen (wenn man den Fall $k^\nu b_\nu = 0$ nicht mit in Betracht zieht). Da aber die b_ν q-Zahlen sein sollen, weiß man nicht recht, was (13) bedeuten soll. Das wird im nächsten Abschnitt klarer werden.

4. – Eine weitere Umformung der Gleichung (2).

Zur weiteren Diskussion des Inhalts der Gl. (2) ist eine weitere Umformung nützlich. Wir gehen noch einmal zurück auf Gl. (9) und schreiben sie so:

$$\partial_\nu \partial^\nu \Phi(y) + \partial_\nu \Phi(y) b^\nu = (\partial_\nu \partial^\nu + \partial_\nu \overleftarrow{b^\nu}) \Phi = \left(\partial_\nu + \frac{\overleftarrow{b_\nu}}{2}\right)\left(\partial^\nu + \frac{\overleftarrow{b^\nu}}{2}\right)\Phi - \frac{\overleftarrow{b^\nu}\overleftarrow{b_\nu}}{4}\Phi; \quad \text{d. h.}$$

$$\left[\left(\partial_\nu + \frac{\overleftarrow{b_\nu}}{2}\right)\left(\partial^\nu + \frac{\overleftarrow{b^\nu}}{2}\right) - \lambda^2\right]\Phi = 0\,, \tag{15}$$

mit

$$\lambda^2 \equiv \frac{\overleftarrow{b_\nu}\overleftarrow{b^\nu}}{4}; \qquad \overleftarrow{b_\nu}\Phi \equiv \Phi b_\nu; \qquad \overleftarrow{b^\nu}\overleftarrow{b_\nu}\Phi \equiv \Phi b^\nu b_\nu\,.$$

Gl. (15) kann man leicht auf eine bekannte Form bringen, indem man setzt:

$$\Phi(y) = \Phi^{(0)}(y) \exp\left[-\frac{b^\nu y_\nu}{2}\right], \tag{16}$$

wobei $\Phi^{(0)}$ der Gleichung

$$(\partial_\nu \partial^\nu - \lambda^2)\,\Phi^{(0)}(y) = 0; \tag{17}$$

genügen müßte. Die Gleichwertigkeit von (16) und (17) mit (15) erkennt man, wenn man die Relation

$$\partial_\nu \Phi(y) = (\partial_\nu \Phi^{(0)}(y)) \exp\left[-\frac{b^\lambda y_\lambda}{2}\right] + \Phi^{(0)}(y)\left(\frac{-b_\nu}{2}\right) \exp\left[-\frac{b^\lambda y_\lambda}{2}\right], \tag{18}$$

verwendet. Gl. (17) ist formal die Feldgleichung eines skalaren Materiefeldes für Teilchen der Masse $\sqrt{\lambda^2}$ (in den üblichen Einheiten $\hbar = 1$; $c = 1$). Neu ist natürlich, daß λ^2 hier eine q-Zahl darstellt. Die Eigenwerte von $\sqrt{\lambda^2}$ haben demnach irgendetwas mit Massen-Eigenwerten zu tun.

Das ist ein vielversprechender Aspekt, und dieses Resultat bestätigt eine Hoffnung, die der Autor schon seit einiger Zeit hegt, daß man nämlich bei gebührender Berücksichtigung der Beschränkungen der Meßbarkeit von Feldern die Möglichkeit erhält, ein Massenspektrum zu berechnen. Wenn wir über die ξ_μ bzw. $K_{\mu\nu}$ bzw. b_ν genügend wüßten, könnten wir ja tatsächlich die Eigenwerte von $b_\nu b^\nu$ berechnen. Aber soweit sind wir im Augenblick noch nicht. (Vor allem wäre vorher natürlich eine Übertragung aller obigen Entwicklungen auf den Fall eines Spinorfeldes erforderlich.) Wir möchten es im Moment sogar noch offen lassen, ob die Eigenwerte von $\sqrt{\lambda^2}$ etwas mit den

Massen *realer* Teilchen zu tun haben; es wäre ja auch denkbar, daß es sich nur um « virtuelle Zustände » des in der konventionellen Theorie allein vorhandenen Teilchens der Masse 0 handelt. In dieser Arbeit wollen wir uns versuchsweise auf den letztgenannten Standpunkt stellen.

5. – Ein einfaches Modell mit nur 2 Zuständen verschiedener Masse.

Der einfachst-mögliche nicht-triviale Fall dürfte vorliegen, wenn das wirklich vorhandene Teilchen die Möglichkeit hat, zwei verschiedene Massen zu zeigen. Man könnte vielleicht auch sagen: Die Masse des betr. Teilchens ist etwas unscharf. Daher auch: Das reale Teilchen « springt » zwischen den beiden möglichen Massen « hin und her » (was natürlich nicht wörtlich zu verstehen ist) (*). Dieser Konzeption entsprechend setzen wir an:

$$\lambda^2 = \begin{pmatrix} \lambda_{00}^2 & 0 \\ 0 & \lambda_{11}^2 \end{pmatrix}; \qquad \Phi^{(0)}(y) = \begin{pmatrix} \Phi_{00}(y) & 0 \\ 0 & \Phi_{11}(y) \end{pmatrix}. \tag{19}$$

λ_{00}, λ_{11} seien zwei beliebige Zahlen, die Größen Φ_{00} bzw. Φ_{11} befriedigen dann die Gleichungen:

$$(\Box - \lambda_{00}^2)\,\Phi_{00} = 0\,; \tag{20a}$$

$$(\Box - \lambda_{11}^2)\,\Phi_{11} = 0\,. \tag{20b}$$

(Keinen rechten Sinn gibt es, wenn man λ^2 und $\Phi^{(0)}$ als nicht-vertauschbar ansieht.) $\Phi_{00}(y)$ und $\Phi_{11}(y)$ sind natürlich noch Operatoren, und es erscheint dem Autor sachgemäß, sinnvoll und auf jeden Fall am einfachsten, sie zunächst versuchsweise (jedes der beiden Felder für sich) wie ein konventionelles Feld für Teilchen der Masse $\sqrt{\lambda_{00}^2}$ bzw. $\sqrt{\lambda_{11}^2}$ zu quantisieren.

Natürlich muß die Theorie so beschaffen sein, daß die beiden Operatoren $\Phi_{00}(y)$ und $\Phi_{11}(y)$ für sich keine physikalische Bedeutung besitzten. Nur die Größe $\Phi(y)$ gemäß (16) hat die Bedeutung eines Feldes im Rahmen unserer Theorie.

Allerdings muß man, um nicht auf die konventionelle Theorie zurückzufallen und der Theorie vernünftige Eigenschaften zu sichern, noch 2 Verfeinerungen anbringen. Die erste ist einfach eine Symmetrisierung: Wir haben ja in (2) die Reihenfolge von Φ und $K^{\mu\nu}$ ganz willkürlich festgesetzt. Sicher

(*) Eine solche Verdoppelung des Hilbertraumes müßte natürlich dadurch hervorgerufen sein, daß die mit den ξ_μ-Operatoren verknüpften Probekörper in zwei verschiedenen quantenmechanischen Zuständen auftreten können.

wäre es vernünftiger, an Stelle von (9) zu setzen:

$$\Box\Phi + \tfrac{1}{2}\{b_\mu \partial^\mu \Phi + (\partial^\mu \Phi) b_\mu\} = 0 . \tag{21}$$

Man kann diese Gl. mit Vorteil auch so schreiben:

$$(D_\mu D^\mu - \lambda^2)\,\Phi = 0 , \tag{21a}$$

mit $D_\mu = \partial_\mu + \frac{1}{4}(b_\mu + \overleftarrow{b}_\mu)$ und

$$\lambda^2 = \tfrac{1}{16}\,(b_\mu b^\mu + \overleftarrow{b}_\mu \overleftarrow{b^\mu} + 2 b_\mu \overleftarrow{b^\mu}) . \tag{22}$$

Dann erkennt man nämlich, daß der Ansatz

$$\Phi(y) = \exp\left[-\frac{b_\nu y^\nu}{4}\right] \Phi^{(0)}(y) \exp\left[-\frac{b_\nu y^\nu}{4}\right], \tag{23}$$

zweckmäßig ist, weil dann

$$(\Box - \lambda^2)\,\Phi^{(0)}(y) = 0 \tag{24}$$

gilt.

Die zweite Verfeinerung betrifft folgenden Umstand: Bei der Zerlegung von $\Phi(y)$ gemäß (23) wird ja der Punkt $y = 0$ sehr stark ausgezeichnet, denn es gilt nach (23) $\Phi(0) = \Phi^{(0)}(0)$. Das hat gewisse unerfreuliche Konsequenzen, denen man entgehen kann, weil ja die Zerlegung von Φ nach (23) mit (24) auch dann gültig bleibt, wenn man statt (23) schreibt:

$$\Phi(y) = \varphi(y)\,\Phi^{(0)}(y)\varphi(y) , \tag{23a}$$

mit (24) und

$$\partial_\nu \varphi(y) = -\frac{b_\nu}{4}\,\varphi(y) . \tag{25}$$

Die allgemeine Lösung von (25) ist aber:

$$\varphi(y) \quad \exp\left[-\frac{b^\nu y_\nu}{4}\right] \cdot C , \tag{26}$$

(*)

(*) Damit dies eine Lösung von (25) ist, müssen allerdings die b^ν alle vier untereinander kommutieren. Dies folgt auch zwangsläufig aus (11) und (14). Hierauf hat mich freundlicherweise Herr Dr. A. Uhlmann, Leipzig, hingewiesen.

wo G ein beliebiger, von y unabhängiger Operator ist, der mit den b^{ν} kommutiert aber mit $\Phi^{(0)}$ nicht zu kommutieren braucht. Oben war es als sachgemäß oder mindestens natürlich bezeichnet worden, die Operatoren Φ_{00} und Φ_{11} wie freie Felder zu quantisieren. D.h. diese Operatoren erzeugen oder vernichten jeweils ein Teilchen der betreffenden Masse. Wir halten es aber aus Gründen der Korrespondenz für richtig, die Annahme einzuführen, daß auch Φ selbst ein Operator ist, der entweder ein Teilchen erzeugt oder ein Teilchen vernichtet. (Nur die Masse des Teilchens ist bei Verwendung von Φ nicht exakt festgelegt.) Dann folgt natürlich, daß der Operator φ nach (26) die Teilchenzahl nicht verändern darf. φ darf aber z.B. ein Teilchen von einem in einen anderen Massenzustand befördern. Das muß es sogar, wenn unsere Theorie mehr als eine triviale Überlagerung zweier Felder verschiedener Masse sein soll.

6. – Die einfachste Green-Funktion.

Die Funktion $\Delta^{(+)\prime}$ ist definiert durch

$$i\Delta^{(+)\prime}(y', y) = \langle \Phi(y')\, \Phi(y) \rangle_0 , \tag{27}$$

vergl. z.B. [9] oder [10]; $\langle\ \rangle_0$ ist der Vakuum-Erwartungswert. Unter Einführung eines Systems von 1-Teilchenzuständen w_k und des Vakuum-Zustands w_0 kann man in bekannter Weise schreiben:

$$i\Delta^{(+)\prime}(y', y) = \sum_{k_1 k''} (w_0, \Phi^{(0)}(y') w_{k''})\, \zeta(w_k, \Phi^{(0)}(y) w_0) , \tag{28}$$

mit (23a) $\varphi w_0 = w_0$, (*) (26) und

$$\zeta = \left(w_{k''}, \exp\left(-\frac{b^{\nu} y'_{\nu}}{4}\right) G \exp\left(-\frac{b^{\nu} y_{\nu}}{4}\right) G w_k \right).$$

Hier wurde neben dem Vollständigkeitstheorem die im Abschnitt **5** formulierte Annahme über die Matrixelemente von $\Phi(y)$ benutzt. (Man muß natürlich beachten, das $\sum_k$ jetzt wegen (19) (2 Massenzustände!) doppelt soviel Glieder hat wie in der konventionellen Theorie des Feldes $\Box U = 0$).

([9]) H. Lehmann: *Nuovo Cimento*, **11**, 342 (1954).

([10]) G. Heber: *Vorlesungen über ausgewählte Kapitel der Quantenfeldtheorie* (Berlin, 1961), S. 16.

(*) Das ist wieder eine willkürliche, aber möglichst einfache Festsetzung!

Um ζ angeben zu können, wollen wir die Wirkung des Operators $\exp[-b^\nu y_\nu/4]G$ auf w_k festlegen; es sei

$$\exp\left[-\frac{b^\nu y_\nu}{4}\right] G w_k = \sum_{k'} g(k, k') \exp[i(k-k')y] \cdot w_{k'} \qquad (*) . \tag{29}$$

$g(k, k') = (w_{k'}, G w_k)$ ist hierbei noch weitgehend beliebig. Eine kurze Rechnung liefert:

$$\zeta = \sum_{k'} g(k, k')\, g(k', k'') \exp[i(k-k')y] \exp[i(k'-k'')y'] ,$$

folglich

$$i\Delta^{(+)\prime}(y', y) = \sum_{k'} \delta(k'^2 - \lambda^2)\, \tilde{g}(k') \exp[-ik'(y-y')] \quad (**) , \tag{30}$$

mit

$$\tilde{g}(k')\, \delta(k'^2 - \lambda^2) = \sum_{k,k''} g(k, k')\, g(k', k'')\, \delta(k''^2 - \lambda^2)\, \delta(k^2 - \lambda^2) \quad (**) . \tag{31}$$

Die Summe über k' enthält natürlich zu jedem λ-Eigenwert eine unendliche k'-Summe, die genau die konventionelle, zu dem betreffenden λ-Eigenwert gehörige $\Delta^{(+)}$-Funktion wäre, wenn $g(k) = 1$ gelten würde. Das wäre dann eine triviale Überlagerung von Feldern verschiedener Masse, denn $g(k) = 1$ führt auf $b^\nu = 0$ zurück. Der einfachste nicht-triviale Fall dürfte es sein, wenn λ^2 zwei Eigenwerte, λ_{00}^2 und λ_{11}^2, besitzt und wenn

$$g^2(k) = \begin{cases} c_0 & \text{für} \quad k^2 = \lambda_{00}^2 , \\ c_1 & \text{für} \quad k^2 = \lambda_{11}^2 , \end{cases}$$

gilt. Man hat dann die einfachste Form einer genau nach Pauli-Villars [11] regularisierten $\Delta^{(+)}$-Funktion, wobei aber diese Regularisierung nicht mehr als formal, sondern als durchaus physikalisch begründet zu bezeichnen wäre (***).

Sie ist eben eine Folge der Massen-Fluktuation, der Unschärfe von λ^2.

(*) Die y-Abhängigkeit der rechten Seite von (29) wird in dieser Form durch Betrachtung der y-Abhängigkeit der einfachsten Matrixelemente von $\Phi^{(0)}(y)$ und $\Phi(y)$ nahegelegt.

(**) Hier ist die Abkürzung

$$\delta(k'^2 - \lambda^2) \equiv \delta(k'^2 - \lambda_{00}^2) + \delta(k'^2 - \lambda_{11}^2) ,$$

eingeführt worden.

(11) W. Pauli und F. Villars: *Rev. Mod. Phys.*, **21**, 434 (1949).

(***) Hierbei müssen wir natürlich die Pauli-Villars-Bedingungen

$$c_0 + c_1 = 0; \qquad c_0 \lambda_{00}^2 + c_1 \lambda_{11}^2 = 0 ,$$

erfüllt denken. Dies setzt voraus, daß G in (26) ein nicht-hermitischer Operator ist; doch steht dem nichts im Wege.

Es wäre hier deshalb auch nicht sinnvoll, $\lambda_{11}^2 \to \infty$ gehen zu lassen, eher würde man $\lambda_{11}^2 \approx \lambda_{00}^2$ als sinnvolle Approximation anzusehen haben.

Will man, daß die $\Delta^{(+)\prime}$-Funktion auch vernünftige (stetige) Ableitungen hat, muß man natürlich mehr als 2 Massen-Zustände in Betracht ziehen. Aber das ist ganz natürlich, weil die obige Annahme, die Probekörper könnten nur in zwei verschiedenen quantenmechanischen Zuständen existieren, selbstverständlich gekünstelt ist und nur eingeführt wurde, um zunächst einen einfachen Überblick zu erhalten.

Eine Indefinitheit von $\tilde{g}(k)$ wie in (32) täuscht übrigens auch eine indefinite Metrik im HR vor, deren Ursprung aber hier in den Matrixelementen von b^ν und G liegt.

Es ist klar, daß der Formel (30) ganz analoge Bildungen auch für die anderen Greenschen Funktionen gelten werden; insbesondere Δ_F' erscheint also ebenfalls in regularisierter Gestalt. Es sieht demnach ganz so aus, als sei es auf diesem Wege möglich, eine konvergente Theorie aufzubauen. Allerdings ist noch nicht klar, ob die hier durch ihre Matrixelemente charakterisierten Operatoren b^ν und G auch wirklich vernünftigen $K^{\mu\nu}(y)$- oder gar $\xi_\mu(y)$-Operatoren entsprechen. Die Klärung dieser Frage dürfte den Rahmen dieser Arbeit sprengen.

Die oben erhaltenen Resultate (insbesondere, daß sich die Existenz der ξ_μ-Operatoren neben den Feldoperatoren als Überlagerung von mehreren Feldern verschiedener Massen äußern kann) sind nach Ansicht des Verf. besonders interessant im Hinblick auf die jüngsten Versuche verschiedener Autoren [12], zu einer Theorie der Elementarteilchen durch Einführung eines entarteten Vakuum-Zustandes usw. vorzustoßen. Wir gelangen ja hier ganz zwangsläufig zu einem erweiterten Hilbertraum.

* * *

Verf. ist Herrn Prof. Marx, Budapest, sehr verbunden für den Hinweis auf die unter [12] zitierten Arbeiten.

Der Autor dankt ferner den Herren Dr. A. Uhlmann und Dipl.-Phys. Chr. Böhme, Leipzig, für nützliche Diskussionen.

Anhang

Wir erwähnten in der Einführung zur vorliegenden Arbeit, daß die Operatoren $\xi_\mu(y)$ bzw. $K_{\mu\nu}(y)$ durch unsere bisherigen Modellbetrachtungen nicht ausreichend fixiert sind. Wir haben uns deshalb oben auf die Frage konzentriert,

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[12] R. E. Marshak und S. Okubo: *Nuovo Cimento*, **19**, 1226 (1961); und dort zitierte weitere Literatur.

ob es denn überhaupt *irgendwelche* Operatoren $K_{\mu\nu}(y)$ gibt, die die Feldgleichungen (2) zur Grundlage einer vernünftigen, nichttrivialen Verallgemeinerung der konventionellen Theorie machen.

Aber es ist natürlich nicht schön, daß die $K_{\mu\nu}(y)$ dabei so weitgehend willkürlich bleiben. Deshalb hat der Autor auch versucht, die $K_{\mu\nu}$ auf einem freilich recht formalen Wege versuchsweise zu bestimmen.

Die Gleichungen (2) folgen ja als Eulersche Gleichungen des Variations-Problems

$$\delta\int \mathscr{L}\cdot \mathrm{d}^4y = 0\,, \tag{A-1}$$

mit

$$\mathscr{L} = -\tfrac{1}{2}\big(\partial_\mu\Phi(y)\big)\big(\partial_\nu\Phi(y)\big)K^{\mu\nu}(y)\,, \tag{A-2}$$

wenn man dieses $\mathscr{L}$ nach Φ variiert. Variation von (A-1) nach K gibt kein brauchbares Resultat. Man könnte aber glauben, daß man durch entsprechende Ergänzung von $\mathscr{L}$ um geeignete Glieder in $K_{\mu\nu}$ ein $\mathscr{L}$ erhalten kann, welches bei Variation nach den $K_{\mu\nu}$ zu einer vernünftigen Gleichung für $K_{\mu\nu}(y)$ führt. Der einfachste derartige Ansatz dürfte sein

$$\mathscr{L} = \mathscr{L}_\Phi + \mathscr{L}_K \tag{A-3}$$

mit $\mathscr{L}_\Phi$ gemäß $\mathscr{L}$ in (A-2) und

$$\mathscr{L}_K = \varkappa\big(K^{\mu\nu}(y) - \delta^{\mu\nu}\big)\big(K_{\mu\nu}(y) - \delta_{\mu\nu}\big); \tag{A-4}$$

$\varkappa$: Parameter.

$\mathscr{L}_K$ ist so gewählt, daß aus $\delta\int \mathscr{L}_K\,\mathrm{d}^4y = 0$ die konventionelle Lösung

$$K^{\mu\nu}(y) = \delta^{\mu\nu} \tag{A-5}$$

folgt. (A-1) mit (A-3) aber liefert:

$$K^{\mu\nu}(y) - \delta^{\mu\nu} = \frac{1}{\varkappa}\big(\partial^\mu\Phi(y)\big)\big(\partial^\nu\Phi(y)\big)\,. \tag{A-6}$$

Man erkennt, daß die Abweichung des $K^{\mu\nu}$ von $\delta^{\mu\nu}$ um so größer ist, je kleiner $\varkappa$ gewählt wird und je rascher sich das Φ-Feld mit y ändert. Man sieht ferner, daß durch (A-6) $K^{\mu\nu}$ völlig eliminiert werden kann, wobei eine in komplizierter Weise nichtlineare Theorie des Φ-Feldes entsteht. Daß die Größen $K_{\mu\nu}$ in dieser Gleichung für $\Phi(y)$ überhaupt nicht mehr vorkommen, ist jedoch für unsere Bestrebungen sehr unerfreulich. Wir waren ja gerade davon ausgegangen, daß neben den $\Phi(y)$ noch andere Operatoren $\xi_\mu(y)$ in der Theorie vorkommen sollten. Ihr Verschwinden führt ja auf eine konventionelle nichtlineare Theorie zurück. Eine entsprechende Schwierigkeit wird natürlich auch dann auftreten, wenn man die Abhängigkeit des $\mathscr{L}_K$ von $K_{\mu\nu}$ komplizierter macht. Es bleibt nur der Ausweg, in (A-4) die $\delta^{\mu\nu}$ durch Opera-

toren $\tilde{\delta}^{\mu\nu}$ zu ersetzen, also

$$\mathscr{L}_K = \varkappa(K^{\mu\nu}(y) - \tilde{\delta}^{\mu\nu})(K_{\mu\nu}(y) - \tilde{\delta}_{\mu\nu}) ,$$

statt (A-4) zu schreiben. Dann sind aber die $\tilde{\delta}^{\mu\nu}$ durch die (A-6) entsprechenden Gleichungen

$$K^{\mu\nu}(y) - \tilde{\delta}^{\mu\nu} = \frac{1}{\varkappa}(\partial^\mu \Phi(y))(\partial^\nu \Phi(y)) , \tag{A-7}$$

nicht bestimmt und wir müssen irgendwelche Annahmen über diese $\tilde{\delta}^{\mu\nu}$ machen. Damit ist die Situation natürlich gar nicht gebessert, denn wir haben jetzt zwar eine nicht unplausible Rückwirkung des Φ-Feldes auf das K-Feld berücksichtigt, aber die Theorie ist wegen der Unbestimmtheit der $\tilde{\delta}^{\mu\nu}$ wiederum nicht abgeschlossen. Die Einführung von mehr oder weniger plausiblen, zweckmäßigerweise aber recht einfachen Annahmen über die $K_{\mu\nu}(y)$ ist nach Ansicht des Autors deshalb im gegenwärtigen Zustand der Theorie unvermeidlich.

RIASSUNTO (*)

Si dimostra che le equazioni di campo per un campo non localizzabile di massa zero (dedotte in [3]) hanno (fra molte altre) soluzioni le cui funzioni di Green sono identiche alle funzioni regolarizzate di Pauli e Villars [11]. Le masse di compensazione hanno qui, tuttavia, un significato fisico in quanto corrispondono alle varie possibili masse di una particella di massa indeterminata. Risulta pertanto dal presente lavoro che dovrebbe essere possibile creare una teoria di campo convergente introducendo nella stessa una massa indeterminata (ad es., entro un piccolissimo intervallo) della particella del campo, o (ciò in questo speciale esempio equivale) posizioni indeterminate dei corpi di prova utilizzati nella misura del campo.

(*) *Traduzione a cura della Redazione.*

On the Triple Angular Correlation and Time Reversal Invariance in Strong Interaction.

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(ricevuto il 27 Luglio 1961)

Summary. — The analysis started in a preceding paper on the angular correlation β-γ-γ from not oriented nuclei, is carried on by examining the case where the first γ is mixed. The simple correlation of the three directions leads to a test for the time reversal in the strong interaction without measuring the polarization of one of the two γ. At the end we consider the example of ^{47}Ca.

1. – Introduction.

In a preceding paper (1) we have given some general formulas for the β-γ-γ correlation and we have seen that if the initial nucleus is not oriented and the γ-polarization is not observed, no conclusion can be drawn as to the time reversal invariance in the weak interaction. In addition, int he direction correlation without any measurement of the polarization, when we deal with pure radiation, we cannot obtain anything definite for the time reversal invariance in the strong interaction. This is due to the fact that the X coefficient is zero when the sum $k_1+k_2+k_3$ is odd, and in order to have a test on the time reversal in strong interaction it is necessary to observe terms with $k_1+k_2+k_3 = \text{odd}$.

On the contrary these terms are present when the first radiation is mixed. In fact limiting our consideration only to the alloved β-transitions, when Gamow-Teller transition is present, there will be $k_1 = 1$, which can be associated to $k_2 = k_3$ even, which gives the aforesaid pseudoscalar term.

(1) V. De Sabbata: *Nuovo Cimento*, **21**, 659 (1961).

Terms with k_2 or k_3 odd will not be present unless the polarization of one or the other radiation or of both, is observed. Therefore to have a test on the time reversal invariance in the strong interaction one simple triple correlation β-γ-γ is sufficient if the first radiation is mixed. The necessity to have a mixed γ is understandable from a physical standpoint as we have to measure a possible phase difference, and this can be obtained as the result of an interference between two coherent radiations.

2. – The first γ mixed.

We can write for the angular correlation function

$$W(\omega_{pr}, \omega_{rs}) = \sum_{k_1 k_2 k_3} \varphi_{k_1 k_2 k_3} \mathscr{Q}^{k_1 k_2 k_3}_{000}(\omega_{pr}, \omega_{rs}) \,, \tag{1}$$

where $\varphi_{k_1 k_2 k_3}$ is the emission factor and $\mathscr{Q}^{k_1 k_2 k_3}_{000}(\omega_{pr}, {}_{rs})$ the angular function as defined in [1]. For the allowed β transition with $\Delta J = |J_2 - J| = 1$, k_1 can assume the values 0 and 1; then we have:

$$\varphi_{k_1 k_2 k_3} = \sum_{rr'} (a^0_{rr'} \omega^0_{rr'} \Phi^{0 k_2 k_3}_{l_2 l_2'} \Phi^{k_3}_{l_2} + a^1_{rr'} \omega^1_{rr'} \Phi^{1 k_2 k_3}_{l_2 l_2'} \Phi^{l_3}_{k_3}) \,, \tag{2}$$

where the $a_{rr'}$ are the usual Fermi and Gamow-Teller coefficients,

$$\omega^0_{00} = \omega^0_{11} = 1; \qquad \omega^1_{01} = \frac{1}{\sqrt{3}} \delta_{J J_2}; \qquad \omega^1_{11} = -\frac{1}{\sqrt{3}} \frac{J_2(J_2+1) - J(J+1) + 2}{2\sqrt{J_2(J_2+1)}} \,,$$

and

$$\Phi^{0 k_2 k_3}_{l_2 l_2'} = \sum_{l_2 l_2'} \hat{J}_2 \hat{J}_3 \hat{l}_2 \hat{l}_2' \hat{k}_2 \hat{k}_3 X \begin{pmatrix} l_2 & J_3 & J_2 \\ l_2' & J_3 & J_2 \\ k_2 & k_3 & 0 \end{pmatrix} (-)^{l_2'-1} \hat{l}_2' (l_2 l_2' 1 -1 \,|\, k_2 0) \gamma_{l_2} \gamma^*_{l_2'} \,, \tag{3}$$

$$\Phi^{1 k_2 k_3}_{l_2 l_2'} = \sum_{l_2 l_2'} \hat{J}_2 \hat{J}_3 \hat{l}_2 \hat{1} \hat{k}_2 \hat{k}_3 X \begin{pmatrix} l_2 & J_3 & J_2 \\ l_2' & J_3 & J_2 \\ k_2 & k_3 & 1 \end{pmatrix} (-)^{l_2'-1} \hat{l}_2' (l_2 l_2' 1 -1 \,|\, k_2 0) \gamma_{l_2} \gamma^*_{l_2'} \,, \tag{4}$$

$$\Phi^{k_3}_{l_3} = \hat{J}_3 \hat{l}_3 (-)^{J_3 + J_3 + l_3 + k_3} \overline{W} \begin{pmatrix} l_3 & k_3 & l_3 \\ J_3 & J_3 & J_3 \end{pmatrix} (-)^{l_3 - 1} (l_3 l_3 1 -1 \,|\, k_3 0) \,|\gamma_{l_3}|^2 \,. \tag{5}$$

Obviously if the radiation is not polarized we have only even values of k_2 and k_3. Odd values of k_2 of k_3 can appear only if one selects respectively the

first or the second circularly polarized γ-ray (with the + or — sign according to left or right polarization).

In order to see when in the correlation there are terms testing the time reversal invariance we discuss briefly the factor:

$$(6) \qquad (-)^{l_3'-1}(l_2\, l_2' 1 -1 \,|\, k_2 0)\, X \begin{pmatrix} l_2 & J_3 & J_2 \\ l_2' & J_3 & J_2 \\ k_2 & k_3 & k_1 \end{pmatrix},$$

that appears in the angular correlation function. If $k_1 = 0$ we must have $k_2 = k_3$; they can assume odd values if one observes the polarization of both γ rays. In the case $k_1 = 1$ for the pure terms (with $l_2 = l_2'$) because of the properties of the X coefficients, only the even values of the $k_1 - k_2 - k_3$ are possible; then one of the k_2, k_3 must be odd. Therefore these terms will be present only if the polarization of the one or the other γ is observed.

For the mixed terms (with $l_2 \neq l_2'$) instead, we can have $k_1 - k_2 - k_3 =$ odd. These terms are therefore present even if the polarization is not observed, in which case both k_2, k_3 are even. When it is summed up on l_2 and l_2' the mixed term will appear twice. Once, as in the expression (6), the second time as in the same expression with l_2 and l_2' interchanged between them; this makes a change of a sign of the expression (6), when $k_1 - k_2 - k_3 =$ odd (and l_2, l_2' have different parity) which leads to the presence of the imaginary part of the product $\gamma_{l_2}\gamma^*_{l_2'}(l_2' \neq l_2)$. On the contrary there will be the real part of $\gamma_{l_2}\gamma^*_{l_2'}$ when $k_1 + k_2 + k_3 =$ even and this will be present only if the polarization of the one or the other γ is observed (where therefore one of the two k_2 or k_3 is odd).

3. – Explicite calculations for the decay of ^{47}Ca nucleus.

The decay scheme is given in Fig. 1. The β-transition is Gamow-Teller allowed and the first γ is mixed $E2$, $M1$. Then it is possible to have terms testing time reversal invariance by measuring only the three directions of motion.

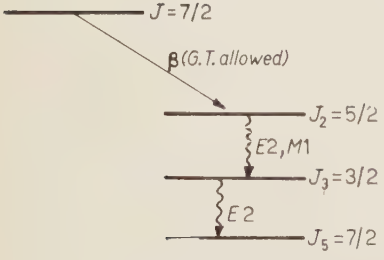

Fig. 1. – Decay scheme of ^{47}Ca.

The β-transition being pure Gamow-Teller, the angular correlation function is reduced to:

$$(7) \qquad W(\omega_{pr}, \omega_{rs}) = \sum_k \Phi^{0kk}_{l_2} \Phi^k_{l_3} \mathscr{D}^{0kk}_{000}(\omega_{pr}, \omega_{rs}) + \\ + A \sqrt{\frac{J-1}{3J}} \sum_{k_2 k_3} \Phi^{1k_2k_3}_{l_2} \Phi^{k_3}_{l_3} \mathscr{D}^{1k_2k_3}_{000}(\omega_{pr}, \omega_{rs}),$$

where:

$$(8) \quad A = \frac{a_{11}^1}{a_{11}^0} = \frac{v}{c} \frac{2\,\mathrm{Re}\,(C_T^* C_T' - C_A^* C_A') - 2\eta\,\mathrm{Im}\,(C_A^* C_T' + C_A'^* C_T)}{|C_A|^2 + |C_A'|^2 + |C_T|^2 + |C_T'|^2 + (2m/E)\mathrm{Re}\,(C_A^* C_T + C_A'^* C_T')},$$

and the Φ are given by (3), (4), (5).

Explicitly in the case of observation of three directions of motions without observing any polarization we have:

$$W(\omega_{pr}, \omega_{rs}) = (1 + |\delta|^2)\Phi_{l_3}^0 + B\Phi_{l_3}^2 P_2(\cos\vartheta_{sr}) + \\ + A\sqrt{\frac{J-1}{3J}}\, C\Phi_{l_3}^2 \sin\vartheta_{pr} \sin\vartheta_{sr} \sin\varphi\, P_2'(\cos\vartheta_{sr}),$$

or

$$(9) \quad W(\omega_{pr}, \omega_{rs}) = 1 + |\delta|^2 - [0.051\,0 - |\delta|\cos\eta\ 0.169 + |\delta|^2\ 0.014\,3] \cdot \\ \cdot (\tfrac{3}{2}\cos^2\vartheta_{sr} - \tfrac{1}{2}) + A \cdot 0.483\,|\delta| \sin\eta \sin\vartheta_{pr} \sin\vartheta_{sr} \sin\varphi \cos\vartheta_{sr},$$

where

$$(10) \quad \delta = \frac{\langle J_2 \| l_2' \| J_3 \rangle}{\langle J_2 \| l_2 \| J_3 \rangle} = |\delta| \exp[i\eta] = \frac{(\|M1\|)}{(\|E2\|)},$$

$$(11) \quad \eta = |\eta_{M1} - \eta_{E2}|,$$

φ is the angle between $\boldsymbol{p} \wedge \boldsymbol{r}$ and $\boldsymbol{s} \wedge \boldsymbol{r}$.

In the numerical calculation of B and C we have used the tables of X-coefficients given by SMITH and STEVENSON ([2,3]) (see Appendix). We see that the term having $\sin\eta$ is a direct test of time reversal. In fact this term does not appear if time reversal invariance holds. The experiment is now in progress at Bologna University.

If in the experiment we select the first circularly polarized γ we have also terms with $\mathscr{D}^{110}$, $\mathscr{D}^{112}$ and $\mathscr{D}^{132}$ (and the same terms with a change of sign for opposite polarization). On the contrary if the polarization of the second γ-ray is observed, there will be terms with $\mathscr{D}^{101}$, $\mathscr{D}^{121}$, $\mathscr{D}^{123}$ and $\mathscr{D}^{143}$.

At last, if the polarization of both γ-rays is observed, there will be also terms with $\mathscr{D}^{011}$, $\mathscr{D}^{033}$ and $\mathscr{D}^{111}$, $\mathscr{D}^{133}$. The expression representing all these cases

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([2]) K. SMITH and J. V. STEVENSON: *Table of Wigner coefficients*, Argonne National Laboratory ANL-5776.

([3]) K. SMITH: *Supplement to a table of Wigner coefficients*, Argonne National Laboratory ANL-5860, part I and II.

can therefore be written as follows:

$$W_1(\omega_{pr}, \omega_{rs}) = W(\omega_{pr}, \omega_{rs}) + P_1^{k_2} A \sqrt{\frac{J-1}{3J}} \cdot$$

$$\cdot \Big\{ C_{11} \cos\vartheta_{pr} \Phi^0_{l_3} + C_{12}[3\cos\vartheta_{rs}\cos\vartheta_{ps} - \cos\vartheta_{pr}]\Phi^2_{l_3} +$$

$$+ C_{13}\left[3\cos\vartheta_{sr}\cos\vartheta_{pr} - \frac{15}{2}\cos^2\vartheta_{sr}\cos\vartheta_{ps} + \frac{3}{2}\cos\vartheta_{ps}\right]\Phi^2_{l_3}\Big\} +$$

$$+ P_2^{k_3} A \sqrt{\frac{J-1}{3J}}\Big\{ C_{21}\cos\vartheta_{ps}\Phi^1_{l_3} + C_{22}[\cos\vartheta_{pr} - 3\cos\vartheta_{rs}\cos\vartheta_{ps}]\Phi^1_{l_3} +$$

$$+ C_{23}[\tfrac{3}{2}(5\cos^2\vartheta_{rs} - 1)\cos\vartheta_{ps} - 3\cos\vartheta_{rs}\cos\vartheta_{pr}]\Phi^3_{l_3} +$$

$$+ C_{24}\left[\tfrac{3}{2}(5\cos^2\vartheta_{rs} - 1)\cos\vartheta_{pr} - \left(\frac{35}{2}\cos^3\vartheta_{rs} - \frac{15}{2}\cos\vartheta_{rs}\right)\cos\vartheta_{ps}\right]\Phi^3_{l_3}\Big\} +$$

$$+ P_1^{k_2} P_2^{k_3}\Big\{ C_{31}\cos\vartheta_{sr}\Phi^1_{l_3} + C_{32}\left[\tfrac{1}{2}(5\cos^3\vartheta_{sr} - 3\cos\vartheta_{sr})\right]\Phi^3_{l_3} +$$

$$+ A\sqrt{\frac{J-1}{3J}}\left[C_{33}\sin\vartheta_{pr}\sin\vartheta_{sr}\sin\varphi\,\Phi^1_{l_3} +\right.$$

$$\left. + C_{34}\sin\vartheta_{pr}\sin\vartheta_{rs}\sin\varphi\left[\tfrac{3}{2}(5\cos^2\vartheta_{sr} - 1)\right]\Phi^3_{l_3}\right]\Big\} .$$

where the C are given in the Appendix and $P_1^{k_2}$, $P_2^{k_3}$ are 0 if k_2, k_3 are even and $+1$ if k_2, k_3 are odd with $+$ or $-$ according to right or left circular polarization.

In the formulas (11) and (12) we have used for the $\mathscr{D}^{k_1k_2k_3}_{000}$ function the relations

$$(13)\quad \mathscr{D}^{1k\,k+1}_{000}(\omega_{pr}, \omega_{rs}) = \mathscr{D}^{1\,k+1\,k}_{0\,0\,0}(\omega_{ps}, \omega_{sr}) =$$

$$= \frac{(-)^k}{\sqrt{(2k+1)(2k+3)(k+1)}}\left[P'_{k+1}(\cos\vartheta_{rs})\cos\vartheta_{ps} - P'_k(\cos\vartheta_{sr})\cos\vartheta_{pr}\right],$$

$$(14)\quad \mathscr{D}^{1kk}_{000}(\omega_{pr}, \omega_{rs}) = i\frac{(-)^k}{\hat{k}\sqrt{k(k+1)}}\boldsymbol{p}\cdot\boldsymbol{s}\wedge\boldsymbol{r}\,P'_k(\cos\vartheta_{rs}) =$$

$$= i\frac{(-)^k}{\hat{k}\sqrt{k\,(k+1)}}\sin\vartheta_{pr}\sin\vartheta_{sr}\sin\varphi\,P'_k(\cos\vartheta_{sr}) .$$

These obtain directly, from the definition of $\mathscr{D}^{k_1k_2k_3}_{000}(\omega_{ps}\omega_{rs})$ as

$$(15)\qquad \mathscr{D}^{k_1k_2k_3}_{000}(\omega_{pr}, \omega_{rs}) = \sum_{x_1x_2} D^k_{0x_1}(\omega_{pr})(-)^{k_1-x_1}\overline{V}\begin{pmatrix} k_1 & k_2 & k_3 \\ -x_1 & 0 & x_2\end{pmatrix} D^{k_3}_{x_30}(\omega_{rs}) .$$

* * *

I would like to express my sincere thanks to Professor P. VERONESI, Dr. C. MARONI and Dr. E. FUSCHINI for valuable discussions on the subject.

APPENDIX

We give here the expressions for B and C that appear in the formulas (9) and (12):

$$B = \hat{J}_2\hat{J}_3\hat{2}\hat{2}\left[(-)\hat{2}\hat{2}(2\,2\,1\,-1\,|\,2\,0)X\begin{pmatrix}2 & J_3 & J_2\\ 2 & J_3 & J_2\\ 2 & 2 & 0\end{pmatrix}+2\,|\delta|\cos\eta\,\hat{2}\hat{1}(2\,1\,1\,-1\,|\,2\,0)\cdot\right.$$

$$\left.\cdot X\begin{pmatrix}2 & J_3 & J_2\\ 1 & J_3 & J_2\\ 2 & 2 & 0\end{pmatrix}+|\delta|^2\hat{1}\hat{1}(1\,1\,1\,-1\,|\,2\,0)X\begin{pmatrix}1 & J_3 & J_2\\ 1 & J_3 & J_2\\ 2 & 2 & 0\end{pmatrix}\right]\frac{1}{\sqrt{5}}=$$

$$= 0.357 - |\delta|\cos\eta\,1.183 + |\delta|^2\,0.100\,,$$

$$C = \hat{J}_2\hat{J}_3\hat{2}\hat{2}\hat{1}\,2\,|\delta|\sin\eta\,\hat{2}\hat{1}(2\,1\,1\,-1\,|\,2\,0)X\begin{pmatrix}2 & J_3 & J_2\\ 1 & J_3 & J_2\\ 2 & 2 & 1\end{pmatrix}\frac{1}{\sqrt{30}} = -|\delta|\sin\eta\,0.231\,.$$

In addition on has $\sqrt{(J-1)/3J} = 0.488$.

In the case of observation of the polarization of the first γ-ray we have the coefficients:

$$C_{11} = \hat{J}_2\hat{J}_3\,\hat{1}\,\hat{1}\left[(-)\hat{2}\;\hat{2}(2\,2\,1\,-1\,|\,2\,0)X\begin{pmatrix}2 & J_3 & J_2\\ 2 & J_3 & J_2\\ 1 & 0 & 1\end{pmatrix}+2\,|\delta|\cos\eta\,\hat{2}\hat{1}(2\,1\,1\,-1\,|\,1\,0)\cdot\right.$$

$$\left.\cdot X\begin{pmatrix}2 & J_3 & J_2\\ 1 & J_3 & J_2\\ 1 & 0 & 1\end{pmatrix}+|\delta|^2\,\hat{1}\;\hat{1}\,(1\,1\,1\,-1\,|\,1\,0)X\begin{pmatrix}1 & J_3 & J_2\\ 1 & J_3 & J_2\\ 1 & 0 & 1\end{pmatrix}\right]\cdot\frac{1}{\sqrt{3}}=$$

$$= 0.537 - 1.04\,|\delta|\cos\eta - 1.02\,|\delta|^2\,,$$

$$C_{12} = \hat{J}_2\,\hat{J}_3\,\hat{1}\;\hat{1}\;\hat{2}\left|(-)\hat{2}\;\hat{2}(2\,2\,1\,-1\,|\,1\,0)X\begin{pmatrix}2 & J_3 & J_2\\ 2 & J_3 & J_2\\ 1 & 2 & 1\end{pmatrix}+2\,|\delta|\cos\eta\,\hat{2}\hat{1}(2\,1\,1\,-1\,|\,1\,0)\cdot\right.$$

$$\left.\cdot X\begin{pmatrix}2 & J_3 & J_2\\ 1 & J_3 & J_2\\ 1 & 2 & 1\end{pmatrix}+|\delta|^2\,\hat{1}\;\hat{1}(1\,1\,1\,-1\,|\,1\,0)X\begin{pmatrix}1 & J_3 & J_2\\ 1 & J_3 & J_2\\ 1 & 2 & 1\end{pmatrix}\right]\left(-\frac{1}{\sqrt{30}}\right)=$$

$$= -0.166 - 0.623\,|\delta|\cos\eta + 0.205\,|\delta|^2\,,$$

$$C_{13} \quad \hat{J}_2\hat{J}_3\,\hat{1}\,\hat{3}\,\hat{2}\left[(-)\hat{2}\,\hat{2}(2\,2\,1\,-1\,|\,3\,0)X\begin{pmatrix}2 & J_3 & J_2\\ 2 & J_3 & J_2\\ 3 & 2 & 1\end{pmatrix}+2\ \delta\ \cos\eta\,\hat{2}\,\hat{1}(2\,1\,1\,-1\,|\,3\,0)\cdot\right.$$

$$\left.\cdot X\begin{pmatrix}2 & J_3 & J_2\\ 1 & J_3 & J_2\\ 3 & 2 & 1\end{pmatrix}\right]\cdot\left(-\frac{1}{\sqrt{105}}\right)=-0.217+0.0561\,|\,\delta\,|\cos\eta\,.$$

In the case of observation of the polarization of the second γ-ray we have the coefficients

$$C_{21}\cdot\ \hat{J}_2\hat{J}_3\,\hat{1}\,\hat{1}\left[(-)\hat{2}\,\hat{2}(2\,2\,1\,-1\ \ 0\,0)X\begin{pmatrix}2 & J_3 & J_2\\ 2 & J_3 & J_2\\ 0 & 1 & 1\end{pmatrix}-\ \delta^{\,2}\hat{1}\ \hat{1}(1\,1\,1\,-1\ \ 0\,0)\cdot\right.$$

$$\left.\cdot X\begin{pmatrix}1 & J_3 & J_2\\ 1 & J_3 & J_2\\ 0 & 1 & 1\end{pmatrix}\right]\frac{1}{\sqrt{3}}=0.567+|\,\delta\,|^2\,0.916\,,$$

$$C_{22}\quad \hat{J}_2\hat{J}_3\,\hat{1}\,\hat{1}\,\hat{2}\left[(-)\hat{2}\,\hat{2}(2\,2\,1\,-1\,|\,2\,0)X\begin{pmatrix}2 & J_3 & J_2\\ 2 & J_3 & J_2\\ 2 & 1 & 1\end{pmatrix}-2\cdot\delta\ \cos\eta\,\hat{2}\,\hat{1}(2\,1\,1\,-1\ \ 2\,0)\cdot\right.$$

$$\left.\cdot X\begin{pmatrix}2 & J_3 & J_2\\ 1 & J_3 & J_2\\ 2 & 1 & 1\end{pmatrix}+|\,\delta\,|^2\hat{1}\ \hat{1}(1\,1\,1\,-1\,|\,2\,0)X\begin{pmatrix}1 & J_3 & J_2\\ 1 & J_3 & J_2\\ 2 & 1 & 1\end{pmatrix}\right]\frac{1}{\sqrt{30}}=$$

$$=-0.115+0.774\,|\,\delta\,|\cos\eta-|\,\delta\,|^2\,0.0917\,,$$

$$C_{23}=\hat{J}_2\hat{J}_3\hat{1}\,\hat{2}\,\hat{3}\left[(-)\hat{2}\,\hat{2}(2\,2\,1\,-1\,|\,2\,0)X\begin{pmatrix}2 & J_3 & J_2\\ 2 & J_3 & J_2\\ 2 & 3 & 1\end{pmatrix}+2\,|\,\delta\,|\cos\eta\,\hat{2}\,\hat{1}(2\,1\,1\,-1\,|\,2\,0)\cdot\right.$$

$$\left.\cdot X\begin{pmatrix}2 & J_3 & J_2\\ 1 & J_3 & J_2\\ 2 & 3 & 1\end{pmatrix}+|\,\delta\,|^2\hat{1}\ \hat{1}(1\,1\,1\,-1\,|\,2\,0)X\begin{pmatrix}1 & J_3 & J_2\\ 1 & J_3 & J_2\\ 2 & 3 & 1\end{pmatrix}\right]\frac{1}{\sqrt{105}}=$$

$$=0.0788-|\,\delta\,|\cos\eta\,0.184+|\,\delta\,|[\,0.0131\,,$$

$$C_{24}=\hat{J}_2\hat{J}_3\hat{1}\ \hat{4}\,\hat{3}\left[(-)\hat{2}\,\hat{2}(2\,2\,1\,-1\,|\,4\,0)X\begin{pmatrix}2 & J_3 & J_2\\ 2 & J_3 & J_2\\ 4 & 3 & 1\end{pmatrix}\right]\cdot\frac{1}{6\sqrt{7}}=-0.0321\,.$$

If at last we observe the polarization of both γ we have also the coefficients:

$$C_{31} = \hat{J}_2\hat{J}_3\hat{1}\,\hat{1}\left|(-)\hat{2}\,\hat{2}(2\,2\,1-1\,|\,1\,0)X\begin{pmatrix}2 & J_3 & J_2\\ 2 & J_3 & J_2\\ 1 & 1 & 0\end{pmatrix}+2\,|\,\delta\,|\cos\eta\;\hat{2}\,\hat{1}(2\,1\,1-1\,|\,1\,0)\cdot\right.$$

$$\left.\cdot X\begin{pmatrix}2 & J_3 & J_2\\ 1 & J_3 & J_2\\ 1 & 1 & 0\end{pmatrix}+|\,\delta\,|^2\hat{1}\,\hat{1}(1\,1\,1-1\,|\,1\,0)X\begin{pmatrix}1 & J_3 & J_2\\ 1 & J_3 & J_2\\ 1 & 1 & 0\end{pmatrix}\right|\left(-\frac{1}{\sqrt{3}}\right).$$

$$= -0.0745 - |\,\delta\,|\cos\eta\,1.59 + |\,\delta\,|^2 0.671\,,$$

$$C_{32} = \hat{J}_2\hat{J}_3\hat{3}\,\hat{3}\left|(-)\hat{2}\,\hat{2}(2\,2\,1-1\,|\,3\,0)X\begin{pmatrix}2 & J_3 & J_2\\ 2 & J_3 & J_2\\ 3 & 3 & 0\end{pmatrix}+2\,|\,\delta\,|\cos\eta\;\hat{2}\,\hat{1}(2\,1\,1-1\,|\,3\,0)\cdot\right.$$

$$\left.\cdot X\begin{pmatrix}2 & J_3 & J_2\\ 2 & J_3 & J_2\\ 3 & 3 & 0\end{pmatrix}\right|\left(-\frac{1}{\sqrt{7}}\right) = 0.383 - |\,\delta\,|\cos\eta\,0.151\,,$$

$$C_{33} = \hat{J}_2\hat{J}_3\hat{1}\,\hat{1}\,\hat{1}\,2\;|\,\delta\,|\sin\eta\,\hat{2}\,\hat{1}(2\,1\,1-1\;\;1\,0)X\begin{pmatrix}2 & J_3 & J_2\\ 1 & J_3 & J_2\\ 1 & 1 & 1\end{pmatrix}\frac{1}{\sqrt{6}} = 0.929\,|\,\delta\,|\sin\eta\,,$$

$$C_{34} = \hat{J}_2\hat{J}_3\hat{1}\,\hat{3}\,\hat{3}\,2\,|\,\delta\,|\;|\sin\eta\,\hat{2}\,\hat{1}(2\,1\,1-1\,|\,3\,0)X\begin{pmatrix}2 & J_3 & J_2\\ 1 & J_3 & J_2\\ 3 & 3 & 1\end{pmatrix}\frac{1}{2\sqrt{21}} =$$

$$= 0.0147\,|\,\delta\,|\sin\eta\,.$$

In all the expressions the last factor derives from the angular function; for the $\Phi^{k_3}_{l_3}$ we have

$$\Phi^{k_3}_{l_3} \propto \hat{J}_3\hat{l}_3(-)^{k_3}\overline{W}\begin{pmatrix}l_3 & k_3 & l_3\\ J_3 & J_5 & J_3\end{pmatrix}\hat{l}_3(l_3\,l_3\,1-1\,|\,k_3\,0)\,.$$

Then:

$$\Phi^0_{l_3} = 1\,,$$
$$\Phi^2_{l_3} = -0.143\,,$$
$$\Phi^1_{l_3} = -0.447\,,$$
$$\Phi^3_{l_3} = 0.0639\,.$$

RIASSUNTO

Si continua l'analisi fatta in un precedente lavoro sulla correlazione β-γ-γ da nuclei non orientati esaminando il caso in cui il primo γ sia misto. Si vede che la semplice correlazione di tre direzioni porta a un « test » per l'invarianza per inversione del tempo nelle interazioni forti senza che sia necessaria la misura della polarizzazione di uno dei due γ. Si considera infine l'esempio del decadimento β-γ-γ del nucleo Ca^{47}.

LETTERE ALLA REDAZIONE

(La responsabilità scientifica degli scritti inseriti in questa rubrica è completamente lasciata dalla Direzione del periodico ai singoli autori)

Leptonic Decay of the Λ^0-Hyperon.

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(ricevuto il 10 Luglio 1961)

Presently, the experimental information on the leptonic decays of the Λ^0-hyperon is very little. Only two cases decaying in the mode $\Lambda^0 \to p + e^- + \bar{\nu}^-$ have been reported so far (**), both by the Berkeley bubble chamber groups (1). Though the expected frequency for this mode in renormalized UFI theory is 1.6% (2), the observed rate is only 0.2%, which is an order of magnitude smaller than the predicted rate. Considering this discrepancy and inadequacy of the data, further experimental information on the subject would be of immense value for a correct estimation of leptonic decay rates.

We report here two cases which appear to be clear examples of leptonic decays of Λ^0-hyperon with e^- as one of the charged secondaries. The events were found amongst 94 decays ($\Lambda^0 \to p + \pi^-$) obtained during a systematic scan for V-events in an emulsion stack exposed to the magnetically separated K^--beam of the Berkeley Bevatron. The details of the exposure and scanning procedure have been described elsewhere (3). Both the V-events consist of only two outgoing tracks, that of a proton and an electron, identified by constant sagitta and $b^* - p\beta$ measurements respectively. No visible primary track could be associated with either of the V-events. The relevant measurements on the two tracks are given in Table I. The photomicrographs of both the events are also shown.

(*) On leave of absence from the Education Department, M. P. Government, Bhopal.

(**) W. E. Humphrey *et al.*: *Phys. Rev. Lett.* **6**, 478 (1961) refer few more events of this type communicated to them privately.

(1) F. S. Crawford, M. Cresti, M. L. Good, G. R. Kalbfleisch, M. L. Stevenson and H. K. Ticho: *Phys. Rev. Lett.*, **1**, 377 (1958); P. Nordin, J. Orear, L. Reed, A. H. Rosenfeld, F. T. Solmitz, H. D. Taft and R. D. Tripp: *Phys. Rev. Lett.*, **1**, 380 (1958); D. A. Glaser: *Strange Particle Decays; Conf. on High Energy Physics* (Kiev, 1959).

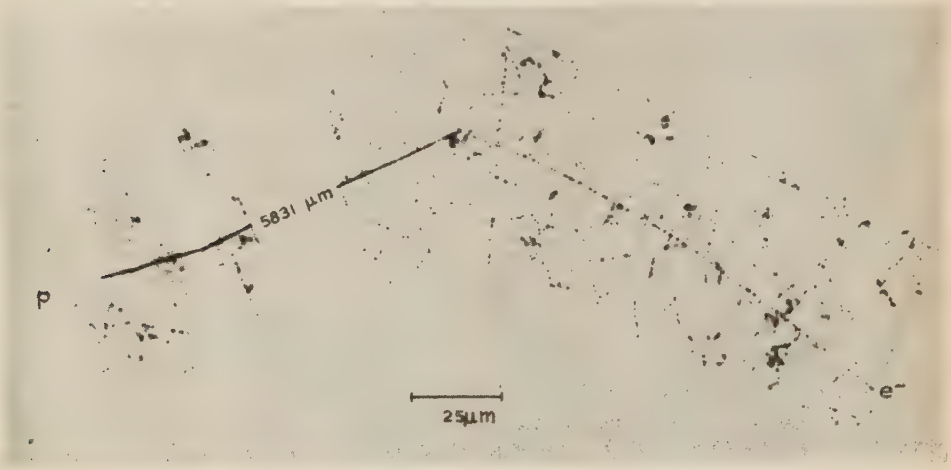
(2) R. P. Feynman and M. Gell-Mann: *Phys. Rev.*, **109**, 380 (1958); S. Okubo, R. E. Marshak and E. C. G. Sudarshan: *Phys. Rev.*, **113**, 944 (1959).

(3) B. Bhowmik, P. C. Jain and P. C. Mathur: *Nuovo Cimento*, **20**, 857 (1961); B. Bhowmik, D. P. Goyal and N. K. Yamdagni: *The mass of the Λ^0-hyperon*, in *Nuovo Cimento* (in press).

TABLE I.

Event no.	Measurements on the proton track			Measurements on the electron track		Space angle between p and e^-
	rest mass	res. range	kinetic energy	b^*	$p\beta$	
	(MeV)	(μm)	(MeV)		(MeV/c)	
1	837^{+409}_{-239}	5 831	38.39	1.11 ± 0.06	26.7 ± 4.1	121° 14′
2	966^{+471}_{-267}	1 071	14.54	1.13 ± 0.08	30.0 ± 6.8	66° 17′

The interpretation that the events are produced by the interaction of a neutral particle with emulsion nucleus where the excited nucleus emits an



Event No. 1.

Event No. 2.

electron along with a proton is excluded because of high observed $p\beta$ of the electrons in both the cases. We therefore assume that the events represent the decay of a neutral particle. The only known unstable neutral particles which on decay can give rise to such events are the neutron and the Λ^0-hyperon. The neutron decay is completely excluded on kinematical grounds. (For the observed energies of the electrons the neutron has to be as fast as $\beta_N=0.999$. The velocity of the proton should then also be $\simeq 0.9$.) We therefore label these events as due to the decay of a Λ^0-hyperon in the mode:

$$\Lambda^0 \rightarrow p + e^- + \bar{\nu}^- .$$

The possibility that the events may be due to the decay of an unknown neutral particle is not excluded.

The events are observed in a sample of 94, $\Lambda^0 \to p + \pi^-$ decays. Taking the ratio $(\Lambda^0 \to p + \pi^-)/(\text{all } \Lambda^0) = \frac{2}{3}$, we get the total number of Λ^0 produced equal to 141. Thus on our statistics the frequency of electronic decay of the Λ^0-hyperon is 1.4% — a value consistent with theoretical prediction. However, if we combine our data with world emulsion data ([4,5]) which report only non-leptonic decays the frequency gets reduced to 0.2%. The true leptonic decay rate can be established only after evaluating relative detection efficiency (*) of leptonic and non-leptonic decay events.

The detection efficiency of

$$\Lambda^0 \to \mu^- + p + \bar{\nu} ,$$

mode is expected to be the same as that of the $\Lambda^0 \to \pi^- + p$ mode. However, a $\Lambda^0 \to \mu$ decay is likely to be classified as Λ-like interaction as the identity of the stopping muon cannot be established by its terminal behaviour which is identical with the terminal behaviour of 30% of stopping pions namely a 0-prong star. Only on careful mass measurement on the lighter partner in the V-event the identity of the muon can be established unambiguously under favourable geometrical conditions (long and flat tracks). The predicted theoretical rate is 0.24% and on a world sample of 1000 V-events a couple of $\Lambda^0 \to \mu$ decays is expected to be observed. No clear example has yet been established. A careful scrutiny of Λ^0-like interactions whose lighter partner produces a star with small visible energy or a 0-prong star may reveal the possible existence of the $\Lambda^0 \to \mu$ decay mode.

* * *

We are indebted to Prof. E. J. LOFGREN of the Berkeley Radiation Laboratory for the exposure facility at the Bevatron. We are grateful to Prof. C. F. POWELL for the processing facility at Bristol. Our thanks are due to Prof. D. J. PROWSE for handling the exposure. One of us (N.K.Y.) thanks C.S.I.R. for the award of a fellowship and the M. P. Govt. for the grant of study leave. Financial support from the Department of Atomic Energy is gratefully acknowledged.

([4]) C. J. MASON, W. H. BARKAS, J. N. DYER, H. H. HECKMAN, N. A. NICKOLS and F. M. M. SMITH: *Bull. Am. Phys. Soc.*, **5**, 224 (1960).

([5]) K. GOTTSTEIN: *Proc. of the Rochester Conference* (1960), p. 436.

(*) *Note added in proof.* – The scanning efficiencies of two observers have been determined by the method of coincidence and are practically the same. The average efficiency for V-events is 91% and for hanging protons ([3]) 89%. Since the detection efficiency for tracks of plateau ionization is known accurately as 94% from the scrutiny of electrons in $\pi \to \mu \to e$ events, the scanning efficiency for $\Lambda^0 \to \beta$ decay becomes 84%. This correction changes our observed leptonic decay rate from 1.4% to 1.6%.

The Y_0^{**}, the π'^0, and the Bound State Model.

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(ricevuto il 14 Luglio 1961)

In a recent note (1) it was pointed out that an S-wave π-Λ interaction strong enough to bind (thus forming the Σ-particle (2)) is consistent with the existence of an $S_{\frac{1}{2}}$ π-Λ resonance, the Y_1^*, at a barycentric total energy of 1385 MeV (3). It is interesting to inquire into the effects of such a strong π-Λ interaction in the π-π-Λ system. In general one expects that a stronger π-Λ interaction is required to bind two π's than one (cf. the strong coupling model for nucleon isobars), and the relative « weakness » of the binding of the π to the Λ leads one to believe that no bound π-π-Λ system exists (the fact that none has been seen also contributes to that belief). That the π-Λ binding is relatively weak can be seen by comparing the « size » of the bound state Σ, which is ~ 1.3 fermi, to the range of the longest-range force contributing to the binding (~ 0.7 fermi associated with 2π exchange). It can also be seen by comparing the π-Λ interaction to the $\bar{K}$-$\mathcal{N}$ interaction, which is so strong as to « bind » both one and *two* $\bar{K}$'s, making the Λ and the Ξ, but with such a small « size » ($\lesssim m_K^{-1}$) relative to the longest-range force between the hypothetical constituents (again the 2π exchange) that the bound state model for these particles is not so immediately useful (and may never be) as it is for the Σ-particle (2,4).

However a « virtual » bound state in the π-π-Λ system would not be surprising, especially if a strong attractive force existed between the pions at low energy. Suppose that such a « virtual » bound state, the Y**, exists at a very low kinetic energy in the barycentric frame, say at ~ 10 MeV (corresponding to a

(*) Partially supported by National Science Foundation Grant G14688.

(**) Partially supported by Office of Naval Research Grant 1677(03).

(1) S. Barshay and H. N. Pendleton: *Phys. Rev. Lett.*, **6**, 421 (1961).

(2) S. Barshay and M. Schwartz: *Phys. Rev. Lett.*, **4**, 618 (1960).

(3) If we redo the estimate of our previous note using the current Dalitz (a-) solution (5), $A_1 = a_1 + ib_1 = -.85 + .21i$ fermi, we obtain a very small positive zero-energy scattering length in the π-Λ system, which in the limit, $b_1 \to 0$, goes to $A \sim +1.4$ fermi, which is quite close to the « size » of the sigma hyperon, ~ 1.3 fermi. However, the computed resonance shape seems to indicate a notably slower fall-off on the high-energy side. We wish to thank Professor Dalitz for sending us his new solutions.

(4) S. Barshay and H. N. Pendleton: *Theory of associated production of sigma hyperons in pion-nucleon collisions* (to be published).

total barycentric energy of ~ 1405 MeV). Then the decay of the Y** into two pions and a Λ would have much less phase space available than the decay in which one of the pions is bound to the Λ (making a Σ), ~ 10 MeV available kinetic energy as compared to ~ 75 MeV.

A possible resonance in the π-Σ system at 1405 MeV [5] may thus be understood in terms of the binding π-Λ interaction and a strong attractive force in the low energy two-pion state. A recent experiment [6] and some theoretical calculations [7] suggest that such a strong two-pion attraction exists at very low energies in the $T=0$ state, but *not* in the $T=1$ or $T=2$ state. Then the π-Σ resonance will be a $T=0$ resonance, as is suggested by experiment [5].

It is attractive to consider the possibility that such a state, centered ~ 27 MeV below the $\bar{\mathrm{K}}$-$\mathcal{N}$ threshold with a half-width of $\sim (10 \div 30)$ MeV, could be responsible for the large $T=0$ absorption cross-section (relative to the $T=1$ absorption) that is observed in the $\bar{\mathrm{K}}$-$\mathcal{N}$ system at rest and at very low energies [8,9]. (It should be recalled that the Y_1^* lies about 47 MeV below the $\bar{\mathrm{K}}$-$\mathcal{N}$ threshold with a half-width of $\sim (10 \div 30)$ MeV [10,11].) In order to be formed in the absorption of a low-energy pseudoscalar $\bar{\mathrm{K}}$ the π-π-Λ state would have to involve the two-pion S-state system in a P-state relative to the Λ. Of course, one or the other of the pions can interact in an S-state with the Λ.

The Dalitz solution [9] for the $T=0$ complex scattering length (corresponding to the (a-) solution), $a_0+ib_0 = -0.75+2i$ fermi, has the correct sign and size for its real part to give rise to a quasi-bound $\bar{\mathrm{K}}$-$\mathcal{N}$ system [9]. However, Dalitz has already noted [9] that the width of this state might be unconfortably large, due to the large value of the imaginary part, b_0. It is assumed that b_0, which is roughly proportional to p^3 [9] (for a π-Σ $P_{\frac{1}{2}}$-state), does not vary *sharply* from its value as determined from data on the $\bar{\mathrm{K}}$-$\mathcal{N}$ system at and above threshold in going to the position of the quasi-bound $\bar{\mathrm{K}}$-$\mathcal{N}$ state. Here p is the relative momentum in the π-Σ barycentric system. However, suppose that the same b_0 corresponds largely to $\bar{\mathrm{K}}+\mathcal{N} \to \pi'^0+\Lambda$ (a quasi-two-body system) where π'^0 is the 2π system at about zero relative kinetic energy. Then in going from the $\bar{\mathrm{K}}$-$\mathcal{N}$ threshold toward the position of the « virtual » bound state $b_0 \to \sim (q/q_t)^n b_0$, where we have assumed $n \sim 3$, and q and q_t are the relative momenta in the τ'^0-Λ system at total barycentric energies of about 1405 MeV and 1432 MeV respectively (*). One assumes for simplicity that the K-matrix element [8] for

$$\bar{\mathrm{K}}+\mathcal{N} \to \pi'^0+\Lambda \,,$$

gives rise to a relatively slow increase in b_0 (as compared to the rapidly decreasing momentum factor) as one goes toward the position of the quasi-two-body « virtual » bound state. In the limit of a

[5] G. GOLDHABER: private communication. We wish to thank Professors S. and G. GOLDHABER for a most stimulating seminar in which the experimental findings of the Alvarez group in Berkeley and of EISENBERG and workers in Israel, concerning the Y_0^{**}, were presented.

[6] A. ABASHIAN, M. E. BOOTH and K. CROWE: *Phys. Rev. Lett.*, **5**, 258 (1960).

[7] B. R. DESAI: *Phys. Rev. Lett.*, **6**, 497 (1961).

[8] R. H. DALITZ and S. F. TUAN: *Ann. Phys.*, **10**, 307 (1960).

[9] R. H. DALITZ: *On the strong interactions of strange particles* (*Rev. Mod. Phys.*, to be published).

[10] M. ALSTON *et al.*: *Phys. Rev. Lett.*, **5**, 520 (1960).

[11] J. P. BERGE *et al.*: *Phys. Rev. Lett.*, **6**, 557 (1961).

(*) A relatively simple model in which b_0 corresponds to the sequence

$$\bar{\mathrm{K}}+\mathcal{N} \to \pi'^0+\Lambda \to \mathrm{Y}_0^{**} \to \pi+\Sigma \,,$$

has been constructed which gives $n \lesssim 4$ (to be published).

quasi-two-body state with $M_{\pi'^0} \sim 280$ MeV

$$b_0 \to (1/8)2 = 0.25 .$$

Consider the Dalitz-Tuan formula [8] for the phase shift δ_0, which will be interpreted as that for the π'^0-Λ P-wave system

$$(1) \qquad \delta_0 = \lambda - \mathrm{tg}^{-1} \frac{k(q/q_t)^3 b_0}{1 - k\,|a_0|} ,$$

where k is the magnitude of the (imaginary) momentum in the $\bar{K}$-$\mathcal{N}$ channel, λ is the phase shift that arises from « dynamics » in the π'^0-Λ system *other* than the coupling to the $\bar{K}$-$\mathcal{N}$ channel at threshold. Let λ be represented by an effective range formula

$$(2) \qquad \mathrm{ctg}\,\lambda = (qa)^{-3} - (qR_0)^{-1} .$$

Take the range parameter $R_0 \sim \frac{1}{2} m_\pi$ and fit a by requiring δ_0 to be $\pi/2$ at a total barycentric energy of ~ 1405 MeV; a is then $\sim 1.28/m_\pi$. This is an extremely reasonable number if it is remembered that it is conjectured that the π'^0-Λ system is forming a quasi-two-body « virtual » bound state by virtue of the strongly attractive π-Λ forces (as well as the π-π forces) that are supposed to give rise to a real bound state, the Σ, whose size is $\sim 1/m_\pi$. In Fig. 1 is exhibited the function $\sin^2\delta_0/q^2$ computed from eq. (1) and (2). One sees a rather narrow resonance falling off somewhat more slowly on the high energy side than on the low. The existence of such a resonance with spin $\frac{1}{2}$ and odd parity relative to the Λ would provide support for the hypotheses of the theory: (*a*) a strong (binding) π-Λ interaction leads to a « virtual » bound state in the π-π-Λ system and (*b*) a strong attractive interaction in the S-wave $T=0$ π-π system at very low energies contributes to that virtual state.

An important point is that a « potential » resonance in the zero-energy S-wave π'^0-Λ system would not be narrow [12]; however, it might be confined by an important coupling to the $\bar{K}$-$\mathcal{N}$ $P_{\frac{1}{2}}$-state.

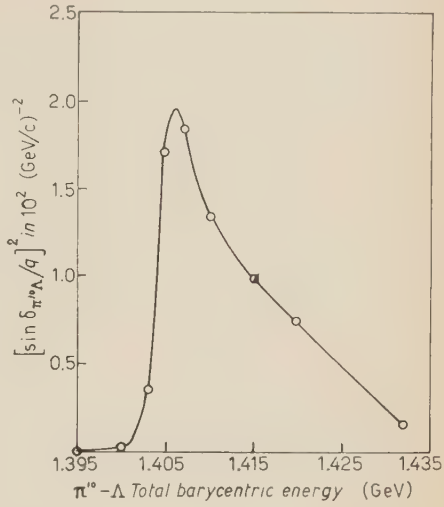

Fig. 1. – The function $\sin^2\delta_0/q^2$ for the π'^0-Λ system computed using formulas (1) and (2) with parameters: $a = 1.28/m_\pi$, $R_0 = 1/2m_\pi$, $b_0 = 2$ fermi, $|a_0| = .75$ fermi, plotted *vs.* total barycentric energy.

Note however, that experience with the coupling of the $P_{\frac{1}{2}}$ π'^0-Λ system to the $\bar{K}$-$\mathcal{N}$ $S_{\frac{1}{2}}$-state (eq. (1)) shows that, in this case, the coupling to $\bar{K}$-$\mathcal{N}$ cannot confine the resonance. Indeed the Y_0^{**}, located within ~ 10 MeV of the π-π-Λ threshold, is *essentially* a result of the quasi-two-body P-wave « virtual » bound state (the π'^0-Λ) and not a result of the coupling to the $\bar{K}$-$\mathcal{N}$ channel. This is to be contrasted with the Y_1^*, which is much closer to the $\bar{K}$-$\mathcal{N}$ threshold than to the π-Λ threshold, with features essentially determined by the coupling

[12] We wish to emphasize that the bound-state model does not necessarily imply that two S-state pions would be bound to a Λ. If the mechanism of the force between pion and Λ involves a change in the nature of the source (such as $\bar{K}$-K exchange, with $\Lambda \to$ nucleon $\to \Lambda$), a second pion would not likely be bound.

to the $\overline{K}$-$\mathcal{N}$ channel (although the precise resonance position and shape may reflect to some extent the size of the strong « potential » interaction in the low energy S-wave π-Λ system [1]; again the « potential » alone cannot confine this system).

Phase space as well as the centrifugal barrier severely suppresses

$$\overline{K} + \mathcal{N} \to \pi + \pi + \Lambda ,$$

near the $\overline{K}$-$\mathcal{N}$ threshold, but one would expect the process $\overline{K} + \mathcal{N} \to 2\pi + \Lambda$ to eventually exhibit a sharp rise (as q^3 for an incident S-wave $\overline{K}$-$\mathcal{N}$ state) to a substantial cross-section (*). (The K-matrix element for the process $\overline{K} + \mathcal{N} \to \pi + \Sigma$ divided by p is assumed to remain relatively large; its value is $B_0 \sim 1/m_\pi^2$ at the $\overline{K}$-$\mathcal{N}$ threshold.) There will be a further variation of b_0 with energy above the $\overline{K}$-$\mathcal{N}$ threshold which will depend upon the behavior of B_0 as well as the phase space and centrifugal barrier factors. This affords yet more room for energy variation in quantities sensitive to b_0 (**).

(*) An important contribution may also come from an incident K-$\mathcal{N}$ $P_{\frac{1}{2}}$ state going to a strongly interacting state of two S-wave pions and a Λ.

(**) It is possible that the 400 MeV/c anomaly in K-$\mathcal{N}$ scattering is due to strong absorption in the isotopic spin zero state, due to the processes

$$K + \mathcal{N} \to \pi^0 + \Lambda \to \pi + \pi + \Lambda ,$$

and

$$K + \mathcal{N} \to \pi + Y_0^{**} \to \pi + \Sigma .$$

These pion-hyperon resonant states could be the $P_{\frac{3}{2}}$ anlogue of the $P_{\frac{1}{2}}$.

Y_0^{**} and would be created by an incident K-$\mathcal{N}$ $D_{\frac{3}{2}}$ state.

Remarks on the Anomaly in Meson Production in p + d Collisions.

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(ricevuto il 17 Luglio 1961)

A good deal of attention has been recently polarized on the production experiment of Abashian, Booth and Crowe [1], made with protons on deuterium. It is known that, at different incident proton energies, the distribution in the recoiling ^{3}He nucleus shows a bump roughly corresponding to a fixed mass of about 300 MeV for the unobserved produced system (or particle). Such an effect cannot be due to the resonant $T=1$, $J=1$ state of two pions, since it is supposed to resonate at much higher energy. The alternative explanations can be some kind of new particle or some enhancement in the production due to a strong (also if not resonant) interaction of the produced pions at the threshold.

The first support of the last point of view was made by Tubis and Uretsky [2] who use a model for introducing the final state interaction. The final state interaction is taken into account in the production cross-section simply by multiplying the phase-space factor by an enhancement factor which is the modulus squared of the final state scattering wave function at a point R out from the interaction region. Then a scattering length approximation is done. They show in this way that a strong interaction at the threshold of the two pions in a P-wave, with a very large P scattering length, can give account of the observed peak at the incoming energy of 743 MeV. The reason why such an effect can simulate a single mass production is made clear from the form of the enhancement factor for the P-wave [3]

$$F \simeq \frac{\lambda^2 q^2}{1 + (q a_1)^6},$$

which depends strongly only on q^2, producing a maximum in the cross-section at various energies at about the same q^2, which is the behaviour of a single mass peak. Obviously the form is different from a single particle peak, buth the width, being actually

(1) A. Abashian, N. E. Booth and K. M. Crowe: *Phys. Rev. Lett.*, **5**, 258 (1960).

(2) A. Tubis and J. L. Uretsky: *Phys. Rev. Lett.*, **5**, 513 (1960).

(3) See for notations ref. (2).

only a function of q^2, behaves at different incident energies as a Breit-Wigner one.

The explanation of a strong threshold P-wave interaction of the two outgoing pions could however not be supported anymore, since from the experimental analysis the assignement $T=1$ was ruled out in favour of a $T=0$ of the produced system.

The problem is then, if the bump can be explained by some threshold interaction of the two π-mesons in as S-state. This question has been partially answered by TRUONG (4), who introduced an exponential potential well for the two-pion interaction and fits the old experimental data at 743 MeV by an S-wave scattering length varying from 0.5 to 1.5 μ^{-1}. The new and more refined experiment at this energy points, however, towards a much higher S-wave scattering length a_0 about 2.5 μ^{-1} ref. (6). Such a scattering length is larger than one would expect, and it can give some trouble in other problems, as the τ-decay and double production experiments, where it seems that such a large a_0 should produce much more relevant $T=0$ effects.

We think thus that the question is still open, for two reasons:

a) It seems always possible to fit the data at one energy by introducing an S interaction of the final pions, with a suitable scattering length.

b) The indication of the 743 MeV experiment for a so high S scattering length needs further confirmation before accepting it, because of the consequences on the before mentioned processes and, for instance, also on the pion-nucleon scattering via the pion-pion nucleon-antinucleon $T=0$ channel.

We think consequently that in order to decide between the two opposite explanations, namely that of a particle (the « three-pion » with $T=0$ and $J=1$, bound state of 3 pion with relative resonating $T=1$, $J=1$ states) and that of a simple final state interaction of of 2 pions in $T=0$, $J=0$, an enrichement is necessary of the experimental data at different incident energies. At just one energy and one fixed laboratory angle, it seems hardly possible to discriminate, within the present experiments.

We have adopted a model derived from that by TUBIS and URETSKY simply substituting an S-wave enhancement factor to that used by them. Our factor is

$$F \simeq \frac{1}{1+(qa_0)^2}.$$

We have tried to see if such a model can be distinguished by its behaviour at various energies from the yield of a single mass peak. The result is that the maximum in the helium momentum distribution will not be in correspondence of a fixed produced mass; however, for the energies at disposal, it is impossible to appreciate this difference, in the p_3 scale.

In Fig. 1 are shown the curves at the incident energy of 743 MeV, for various values of the scattering length. The best fit to the more recent experimental data is given by $a_0=2.5\div2.8$. The fit is not completely good in the region of low p_3, where our theoretical curve lies lower by a 20% with respect to the experimental yield. It is difficult to give reason of such feature. It could perhaps be due to some effective range effect which reinforces the cross-section. No reliable improvement can be however done on the model proposed since, if one takes into account higher effects in q^2, then the factorization of the matrix element in a statistical production amplitude and in a pion-pion final interaction loses its sense, as can be seen, *e.g.* from the analysis by GRIBOV (5).

(4) T. N. TRUONG: *Phys. Rev. Lett.*, **6**, 308 (1961).

(5) J. GRIBOV: *Žurn. Èksp. Teor. Fiz.*, **6**, 1102 (1958).

Connected diagrams give in fact contributions of the same order.

particle whose width, could be about 25 MeV in the p_3 scale.

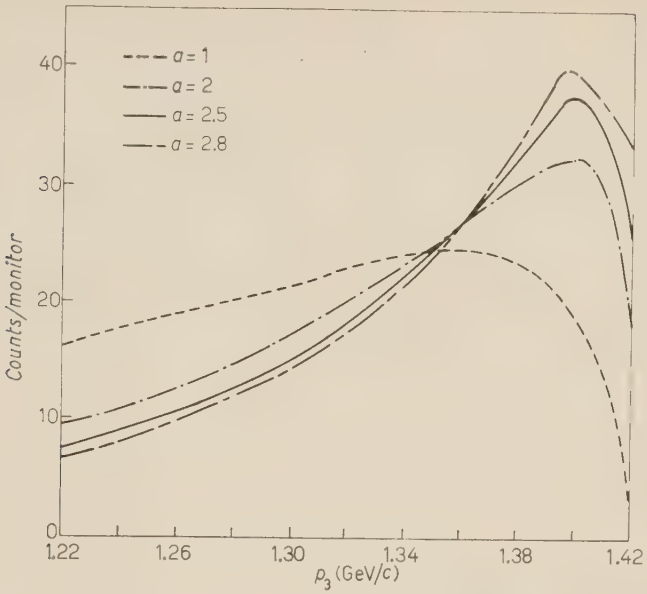

Fig. 1. – Momentum spectrum of ^{3}He at 12° laboratory system and 743 MeV incident proton energy. The curves are calculated for various values of the scattering length and normalized to equal area. The solid curve is the best fit of the experimental data of ref. (6).

It is impossible to distinguish, at that energy, from a peak of a single

In order to see the behaviour at a different energy, in Fig. 2 we have

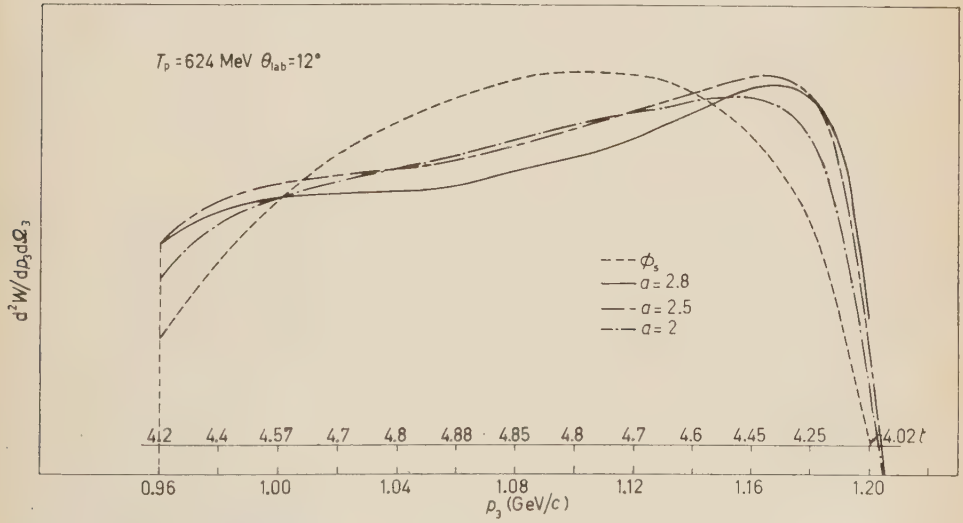

Fig. 2. – Momentum spectrum of ^{3}He at 12° laboratory system and 624 MeV incident proton energy. Curves normalized to equal area. Ordinate in arbitrary units. The curve Φ_s indicates the approximation of simple phase space.

plotted the curves for various values of the scattering length, normalized to the same total area, at the incident energy of 624 MeV. Also shown are the curve obtained by the simple phase space, normalized to the same area as before. It is seen from the figure that it is impossible to observe experimentally that the maximum, in the « enhancing » S-wave case, does not correspond to the same produced mass, since the differences in the p_3 scale are too small to be appreciated. The form of the curve is however completely different. In this way one could discriminate the two possibilities. However we observe that:

peaks (forward and backward production) would mix their contributions in the p_3 spectrum.

Consequently better discrimination between the two possibilities could be obtained by a new experiment at 743 MeV but at another laboratory angle of the recoiling ^{3}He, as pointed out in ABASHIAN, BOOTH and CROWE (6).

For this aim we have done calculations at 15° laboratory angle which seems interesting since, in such condition, in the range of the values of p_3 reasonable for the experimental analysis, the momentum spectrum would show both the peaks of the single particle.

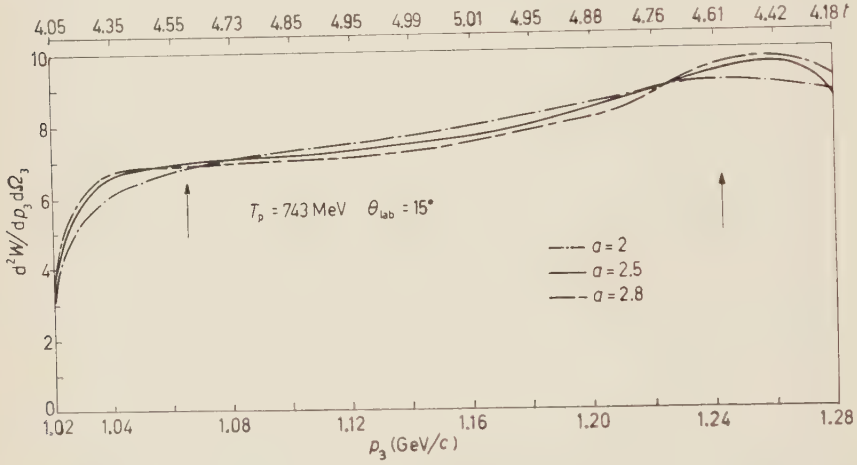

Fig. 3. – Momentum spectrum of ^{3}He at 15° laboratory system and 743 MeV proton incident energy. Curves normalized to equal area. Ordinate in arbitrary units.

1) At that incident energy (624 MeV) the momentum spectrum ranges on values of p_3 near to 1 GeV which makes the experiment very difficult (see ABASHIAN, BOOTH and CROWE (6)).

2) Assuming the width of the particle as given by ABASHIAN, BOOTH and CROWE (1) at 624 MeV the two

Fig. 3 shows the yield of the S-wave model with different scattering length, at 743 MeV and 15° laboratory angle.

Observe that at this angle the variations of the cross-section due to different values of the scattering length are less relevant than at 12°. This fact should allow a clear cut distinction between this interpretation and the effects of the production of a particle of mass around 300 MeV which would give rise to peaks at the positions indicated in the Fig. 3 by the arrows.

(6) A. ABASHIAN, N. E. BOOTH and K. M. CROWE: preprint May 1961.

In conclusion the S-wave model contains a parameter (a_0) which could be determined by means of the best fit of the experimental data at a fixed energy (743 MeV) and a fixed angle (12°). Such a value for the same incident energy but at another angle allows to construct up theoretically a ^{3}He momentum spectrum which, by comparison with the experimental one, should give an unambiguous answer on the validity of the proposed model.

* * *

I am deeply indebted to Dr. V. De Alfaro for help and extremely useful discussions.

I wish to thank warmly Prof. M. Cini for the useful discussions.

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Remarks on the «Effective Target Mass» (*).

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(ricevuto il 19 Luglio 1961)

1. – In interpreting experimental data of pion-nucleon or nucleon-nucleon interactions at high energy a simple model is often employed, in which the nucleon is treated as a composite system of a nucleon-core and a virtual pion surrounding it, and further this virtual pion is usually regarded as if it behaves like a free particle. Such an approach has turned out useful in many cases, at least in a heuristic sense.

Sometimes one dares to go still further, however, and tries to find out directly from experiment the distribution of the «mass» of this virtual particle which takes part in the peripheral collision. Thus the concept of an effective target mass has been introduced by imagining that the incident particle (nucleon or pion, denoted by A) makes a collision not with the whole target nucleon, but with a virtual pion of mass m^* in the cloud, without giving any appreciable influence to the rest of the nucleon. This intuitive but fictitious picture leads, together with the energy-momentum conservation law, to [1]

$$m^* = \sum_i{}' (E_i^{(l)} - p_i^{(l)} \cos\theta_i^{(l)}) , \tag{1}$$

where the superscript (l) denotes a quantity in the laboratory system and $\sum'$ means that the summation is taken over all the secondary particles except the recoil nucleon. Throughout this note we assume $E_A^{(l)} - p_A^{(l)} \ll M, m^*$.

Now we should like to emphasize the necessity of clearly separating the experimental quantity «target mass» defined by the r.h.s. of (1) from its too naive interpretation. We shall see that the target mass (1) should give certain useful information if properly generalized and combined with other quantities, while its straightforward identification with the mass of the virtual pion or an attempt to connect it too literally with the nucleon structure does not appear to be justified.

(*) A full text of this note will appear in the *Acta Phys. Polon.* Preprints are available.
(**) On leave from the Research Institute for Fundamental Physics, Kyoto University.
(1) N. G. Birgen and Yu. A. Smorodir: *Zurn. Eksp. Teor. Fiz.*, **37**, 1355 (1959).

For this purpose we have examined the significance of the « target mass » firstly within the general kinematical framework of the Lorentz transformation and the energy-momentum conservation, and then, more specifically, applying the one-pion-exchange model with the pole approximation.

2. – Kinematical relations.

2·1. – Fig. 1 shows a scheme of production of particles in the collision of an incident particle A (pion or nucleon) with a target nucleon B (mass M). The secondary particles are arbitrarily divided into two groups, the group (II) including the recoil nucleon. The scheme is quite general if we admit that the vertices V_A and V_B can formally include those particles which « go through » without any interaction (*).

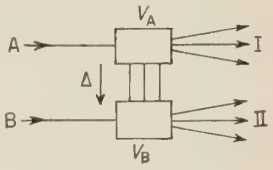

Fig. 1.

A generalized target mass may be defined by

$$m^* = \sum_{i\in(I)} (E_i^{(l)} - p_i^{(l)} \cos\theta_i^{(l)}) , \tag{2}$$

where the summation goes over the particles of the group (I). (Thus the definition of m^* essentially depends on the way of the division.) The original definition (1) is of course included here as a particular case. Obviously we have

$$\sum_{i\in(II)} (E_i^{(l)} - p_i^{(l)} \cos\theta_i^{(l)}) = M - m^* . \tag{3}$$

2·2. – An alternative definition of the target mass, which would be identical with (2) if the intuitive picture were exactly valid, is to introduce the c.m. system of the group (I) and equate it to the c.m. system of « incident particle + a fictitious particle with mass m^{**} at rest in the laboratory system ». We find that the two definitions are related to each other through

$$\frac{m^{**}}{m^*} = \frac{p_i^{(l)}}{\sum_{(I)} p_i^{(l)} \cos\theta_i^{(l)}} , \tag{4}$$

so that they are equivalent as far as $\sum_{(II)} p_i^{(l)} \cos\theta_i^{(l)} \ll p_A^{(l)}$.

2·3. – It is easily seen that

$$m^* \approx \Delta^{(l)} \cos\theta_\Delta^{(l)} - \Delta_0^{(l)} . \tag{5}$$

We define

$$K \equiv m^*/M , \tag{6}$$

(*) As far as purely kinematical considerations are concerned the division into two groups can be quite arbitrary, but it is evident that our arguments may make physical sense only when the division has been done reasonably. Presence of those particles which « go through » the vertices without interaction usually implies that the division has not been made in a proper way.

and it can be shown that K represents the ratio:

$$\frac{\text{energy transferred to the particles of (I) from the particles A and B}}{\text{energy of the particle B}}$$

measured in a coordinate system which is moving with respect to the laboratory system in the direction of the incident particles with such a large Lorentz factor γ that

$$2\gamma^2 M \gg \sum_{(I)} p_i^{(I)} \cos\theta_i^{(I)} .$$

In particular, this Lorentz frame can be identified with the antilaboratory system (mirror system), where the incident and the target particles interchange their roles, and then K is reduced to the partial mirror inelasticity:

$$K = \left(\frac{\text{energy transferred to the group (I)}}{\text{total available energy}}\right)_{\text{mirror}}, \tag{7}$$

a relation, which has been already pointed out and applied by DOBROTIN and SLAVATINSKY [2].

2·4. – In order to express in a concise way the relation of m^* to the invariant energy-momentum transfer $\varkappa^2 \equiv \boldsymbol{\Delta}^2 = \Delta^2 - \Delta_0^2$, we introduce a co-ordinate system (which we shall call S-system), where $\Delta_0^{(S)} = 0$ (*). Denote the Lorentz factor connecting the S-system and the laboratory system by γ_S; then we find

$$m^* \leqslant (\gamma_S - \sqrt{\gamma_S^2 - 1})\varkappa , \tag{8}$$

$$\approx \varkappa/2\gamma_S \qquad (\text{when } \gamma_S \gg 1). \tag{8'}$$

The deviation of the S-system from the c.m. system is a measure of the asymmetry of the collision (**). This can be seen also from the formula,

$$\varkappa^2 \quad 4m^*\gamma^* \left\{ \begin{matrix} m^* \\ 4M\gamma^{*2} \end{matrix} \right. \begin{matrix} W_{\mathrm{I}}^2 - W_{\mathrm{II}}^2 \\ \end{matrix} \Big\} . \tag{9}$$

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[2] N. A. DOBROTIN and S. A. SLAVATINSKY: *Proc. of the 1960 High Energy Conference*, p. 819; Lebedev Institute preprint, A-24 (1960).

(*) This condition alone does not suffice for determining S-system uniquely, because any Lorentz transformation from an S-system — in which the components of Δ are $(\Delta, 0)$ — in the direction perpendicular to $\boldsymbol{\Delta}$ leads to another S-system. This ambiguity, however, does not affect our arguments. But we can fix our S-system for definiteness by imposing an additional condition that it should move with respect to the laboratory system in the direction of the incident particle.

(**) If we apply a specific model (fire-ball or isobar model) we can estimate γ_S for cosmic ray jets with well separated 2-maximum angular distribuiion. The formula for the simplest case is

$$\log \gamma_S = \tfrac{1}{2}\{\log \gamma_{\mathrm{I}} + \log \gamma_{\mathrm{II}} + \log n_{\mathrm{I}} - \log n_{\mathrm{II}}\} ,$$

where $\gamma_{\mathrm{I,II}}$ and $n_{\mathrm{I,II}}$ are the Lorentz factors and the number of shower particles of the fire-balls, respectively.

which expresses x through the target mass, the Lorentz factor of the c.m. system with respect to the laboratory system, γ^*, and the « masses » of the two groups, W_{I} and W_{II}, where $W^2 = (\sum E_i)^2 - (\sum \boldsymbol{p}_i)^2$.

(8) or (9) shows explicitly the essential difference in the magnitude of x and m^*.

3. – Application of the one-pion-exchange model (*) ([3-5]).

3·1. – The OPE-model — which restricts the energy-momentum transfer in Fig. 1 to the exchange of a single pion — has proved to be fruitful in the analysis of pion-nucleon and nucleon-nucleon collisions at a few GeV, while its validity or usefulness seems to become more dubious at higher energies. It should be noticed that this model deviates appreciably from an apparently similar phenomenological model of cosmic ray collisions ([6,7]) in the magnitude of energy-momentum transfer (**).

Nevertheless we are here applying the OPE-model to the analysis of the «target mass » simply because it is the only formalism we can work out at present in a more or less convincing way on the basis of quantum field theory. Even if the model may eventually turn out to be quite unrealistic for superhigh energy events, it will still be instructive to compare the notions of a very intuitive picture with the results of this field-theoretical model.

3·2. – The formula for the distribution of the target mass of the particular division (1) predicted by the OPE-model is

$$(10)\qquad \frac{\partial^4\sigma}{\partial W_{\mathrm{I}}\partial W_{\mathrm{II}}\partial \boldsymbol{\Delta}^2\partial x} = \frac{2}{(2\pi)^3 p_{\mathrm{B}}^{(I)2} l'^2}\,[D_F(\boldsymbol{\Delta}^2)]^2 p_{\mathrm{A}}^{(\mathrm{I})} W_{\mathrm{I}}^2 \cdot \sigma_{\mathrm{A}}(W_{\mathrm{I}}, \boldsymbol{\Delta}^2)\cdot \cdot p_{\mathrm{B}}^{(\mathrm{II})} \cdot W_{\mathrm{II}}^2 \cdot \frac{\partial \sigma_{\mathrm{B}}(W_{\mathrm{II}}, \boldsymbol{\Delta}^2)}{\partial x},$$

where similar notations to reference ([3]) are used and x is an auxiliary scalar variable, $x = p_{\mathrm{A}}^{\nu}\cdot t_{\nu}$, t being the 4-momentum of the recoil nucleon, and is related to m^* through

$$(11)\qquad x \approx -E_{\mathrm{A}}^{(I)}(M - m^*)\,.$$

(*) Abbreviated hereafter as OPE-model.

([3]) F. Salzman and G. Salzman: *Phys. Rev. Lett.*, **5**, 377 (1960); *Phys. Rev.*, **121**, 1541 (1961).

([4]) I. M. Dremin and D. S. Chernavsky: *Žurn. Éksp. Teor. Fiz.*, **38**, 229 (1960); Lebedev Institute preprint, A-44 (1960); I. M. Gramenitsky, I. M. Dremin, V. M. Maksimenko and D. S. Chernavsky: Lebedev Institute preprint (1960).

([5]) F. Bonsignori and F. Selleri: *Nuovo Cimento*, **15**, 465 (1960); F. Selleri: *Phys. Rev. Lett.*, **6**, 64 (1961).

([6]) N. Yajima, S. Takagi and K. Kobayakawa: *Prog. Theor. Phys.*, **24**, 59 (1960).

([7]) E. M. Friedländer: *Phys. Rev. Lett.*, **5**, 212 (1960).

(**) If we estimate the energy-momentum transfer we shall find that most of the cosmic ray peripheral collisions we know at present do not correspond to the legitimate applicability region of the OPE-model. Notice that at very high energy one is obliged to employ pole approximation in the OPE-model in order to get a reasonable cross section. It is possible that those events which are correctly described by the OPE- mechanism do exist but have not been fully detected, because usual cosmic ray experiments are biased against events of small inelasticity and small multiplicity.

The differential cross section $\partial\sigma_B/\partial x$ appearing in (10) can be calculated if we know the angular and momentum distribution of the process at the vertex V_B.

3·3. – As an explicit example we define m^* by the particular division (1) and apply our formula to the case of asymmetric collision of nucleon-nucleon at super-high energy. We put namely $W_{I,II} > M + \mu$, and restrict V_B to «elastic» nucleon-pion scattering, and further make the following simplifying assumptions,

$$\sigma_A(W_I, \Delta^2) = \text{const},$$

$$\sigma_B(W_{II}, \Delta^2) = \begin{cases} \text{const}, & \text{for } a \leqslant W_{II}^2 \leqslant b, \\ 0, & \text{for } W_{II}^2 > b,\ W_{II}^2 < a, \end{cases}$$

where a and b are adjustable parameters. Then we can work out $\partial\sigma/\partial m^*$ as a functional of a and b. Fig. 2 shows two examples: 1) $a = 1.2$ (GeV)^2, $b = 1.3$ (GeV)^2; 2) $a = 3.0$ (GeV)^2, $b = 3.1$ (GeV)^2.

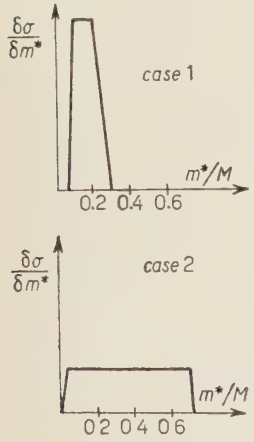

Fig. 2.

From these results we see that the OPE-model does not always lead to a peak in the distribution of the target mass. In this example presence or absence of a peak is connected with properties of the vertex V_B (*). Further, the normalized distribution of m^* does not depend on the incident energy.

Thus we know that we have to be sufficiently careful before deriving a definite conclusion on the nucleon structure from experimental distribution of m^* (**).

3·4. – A particular case, which more or less closely resembles the original intuitive picture of the target mass, is realized when the target nucleon recoils without being excited. Then we find

$$m^* \approx m^{**} \approx \varkappa \cos\theta_r^{(1)}, \tag{13}$$

so that we can get the distribution of $\varkappa$ from that of m^* and recoil angle. The OPE-model predicts, however, that the relative importance of this mechanism is small at several GeV and decreases with energy.

3·5. – Now let us suppose that the division into two groups has been done properly, corresponding to two non-interfering emission centers (**), and study what information we can obtain from experimental knowledge of the target mass.

(*) It is not difficult to construct an example of two-peak distribution: we have only to prescribe an anisotropic angular distribution of pion-nucleon scattering at the vertex V_B.

(**) In fact it is almost obvious that we shall not obtain a sensible result if we are to ascribe all the emitted pions — they are coming out in our example from two distinct vertices — to the effect of a single target mass.

(***) As an example we can imagine a cosmic ray jet shower with well separated inner and outer cones.

For $\varkappa \sim \mu$ and $W > 1.3M$, we get kinematically

$$m^* \approx m^{**} \approx \frac{\varkappa^2 M}{W_{\mathrm{II}}^2 - M^2} - \frac{W_{\mathrm{II}}^2 - M^2 - \varkappa^2}{4M} (\theta_{\Delta}^{(l)})^2 . \tag{14}$$

On the basis of the OPE-model we have

$$\frac{\partial^3 \sigma}{\partial W_{\mathrm{I}} \partial W_{\mathrm{II}} \partial m^*} = \frac{4E_{\mathrm{A}}^{(l)}}{(2\pi)^3 p_{\mathrm{A}}^{(l)} U^2} \cdot \frac{1}{(2E_{\mathrm{A}}^{(l)} m^* - W_{\mathrm{I}}^2)^2} \cdot \tag{15}$$

$$\cdot p_{\mathrm{A}}^{(\mathrm{I})} W_{\mathrm{I}}^2 \cdot \sigma_{\mathrm{A}}(W_{\mathrm{I}}, \Delta^2) \cdot p_{\mathrm{B}}^{(\mathrm{II})} W_{\mathrm{II}}^2 \cdot \sigma_{\mathrm{B}}(W_{\mathrm{II}}, \Delta^2) .$$

In the case of nucleon-nucleon collision

$$\Delta^2 = U^2 \left(\frac{m^*}{M}\right) - M^2 - W_{\mathrm{I}}^2 . \tag{16}$$

For sufficiently small Δ^2 and very large c.m. energy U the allowed region of $(W_{\mathrm{I}}, W_{\mathrm{II}}, \Delta^2)$ is given by

$$U^2 \left[U^2 \left(\frac{m^*}{M}\right) - M^2 - W_{\mathrm{I}}^2 \right] = (W_{\mathrm{I}}^2 - M^2)(W_{\mathrm{II}}^2 - M^2) . \tag{17}$$

From (15) and (16) we see that $\partial\sigma/\partial(m^*/M)$ is a function of $U^2 m^*$, so that the shape of $\partial\sigma/\partial m^*$ depends on energy, the position of eventual maximum tending to zero for increasing U.

Approximate expressions valid for very large U, not too large $\Delta^2_{\max}$ (*), and constant σ_{A}, σ_{B}, are given below.

$$\frac{\partial\sigma}{\partial(m^*/M)} = \frac{[U^2(m^*/M) - 2M^2]}{[U^2(m^*/M) - M^2]} \left\{ \ln \frac{[U^2(m^*/M) - 2M^2]}{2M^3\mu} \cdot \right. \tag{18}$$

$$\left. \cdot [U^2(m^*/M) - M^2] + \frac{2M\mu}{[U^2(m^*/M) - 2M^2]} - 1 \right\} ,$$

$$\begin{cases} (m^*/M)_{\min} \approx \dfrac{M^2 + (M+\mu)^2}{U^2} , \\ (m^*/M)_{\max} \approx \left(\dfrac{2M\mu + \Delta^2_{\max}}{2M^2}\right) \left(\sqrt{1 + \dfrac{4M^2 \Delta^2_{\max}}{(2M\mu + \Delta^2_{\max})^2}} - 1 \right) . \end{cases} \tag{19}$$

* * *

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(*) $\Delta^2_{\max}$ is an additional parameter of the theory.

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Baryon Mass Differences and Isobar States.

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Several attempts have been recently made in order to give a gauge invariant foundation to the strong interactions ([1,2]). In particular, it was proposed to postulate the existence of three different fundamental vector fields, whose gauge properties should be responsible respectively for baryon number, isospin and strangeness conservation in strong interactions, while the well-known pseudoscalar π and K mesons should be considered as complicated compound states of baryons and antibaryons.

This purely theoretical approach seems to have already received some experimental support according to the following data:

a) several features of strong interaction phenomena appear to be satisfactorily interpreted by assuming a pion-pion interaction equivalent to a bipion vector-isovector state $\tau\pi_{\mu j}$ ([3]) ($J=1$, $T=1$) with mass of about $(4 \div 5)\, m_\pi$;

b) from an experiment of Abashian *et al.* ([4]) there seems to be some evidence for the existence of a tripion vector isoscalar state B^0_μ ($J=1$ and $T=0$) with mass of about $2.2\, m_\pi$.

These two vector fields corresponding to two of those postulated by Sakurai ([2]) may be considered as responsible respectively for isospin and baryon number conservation; and quite recently, Bergia *et al.* ([5]), have shown that such an assumption allows a very nice explanation of the nucleon form factors.

For what concerns the third of the fundamental vector fields necessary for the complete gauge invariant approach to the strong interactions, we would like to take as much advantage as possible from the experimental information now available.

([1]) C. N. Yang and R. Mills: *Phys. Rev.*, **96**, 191 (1954); A. Salam and J. C. Ward: *Nuovo Cimento*, **19**, 165 (1961); **20**, 419 (1961); V. Gupta: *Nuovo Cimento*, **18**, 586 (1960).

([2]) J. J. Sakurai: *Ann. Phys.*, **11**, 1 (1960).

([3]) W. R. Frazer and J. R. Fulco: *Phys. Rev. Lett.*, **2**, 365 (1959); E. Pickup, F. Ayer and E. O. Salant: *Phys. Rev. Lett.*, **4**, 474 (1960); J. G. Rushbrooke and D. Radojicic: *Phys. Rev. Lett.*, **4**, 567 (1960).

([4]) A. Abashian, N. E. Booth and K. M. Crowe: *Phys. Rev. Lett.*, **5**, 258 (1960).

([5]) S. Bergia, A. Stanghellini, S. Fubini and C. Villi: *Phys. Rev. Lett.*, **6**, 367 (1961).

Good evidence ([6]) has recently been obtained for a $K\pi$ resonance of about 835 MeV; there are some indications that its isobaric spin is $T=\frac{1}{2}$, while no data concerning its angular momentum are known up to now.

We would like to explore some consequences of the assumption that this $K\pi$ resonance is due to that third vector χ_μ field, which should provide the conservation of hypercharge. It was already shown by C. H. CHAN ([7]), that spin one for such a state would better account for some of the observed cross-sections related to it than spin zero. Moreover, owing to the similarity of the isospin properties, we shall conform for the χ_μ interactions to the same four-dimensional isospace formalism based on the doublet approximation, which has already been developed for the normal K interactions in different previous papers ([8,9]). Such assumptions for this third fundamental vectors differ from those of SAKURAI ([2]).

In the present note, it will be shown that by starting from such a model, a simple physical picture for the symmetries of strong interactions and for the baryon mass differences can be given, leading to a quite similar outlook in the problem as was already proposed by the author ([10]) on purely phenomenological reasons.

Let us first consider the eight baryon states of the doublet approximation as components of a single spinor

$$\Psi = \begin{vmatrix} N \\ \Xi \\ Y_a \\ Y_b \end{vmatrix}, \tag{1}$$

each row referring to a doublet

$$N = \begin{vmatrix} p \\ n \end{vmatrix}, \quad \Xi = \begin{vmatrix} \Xi^0 \\ \Xi^- \end{vmatrix}, \quad Y_a = \begin{vmatrix} -\Sigma^+ \\ Y^0 \end{vmatrix}, \quad Y_b = \begin{vmatrix} Z^0 \\ \Sigma^- \end{vmatrix}, \tag{1a}$$

and the $\pi\pi_{\mu j}$ and $\chi_{\mu K}$ vector interactions to be represented by rotations into two different (respectively three- and four-dimensional) and independent isobaric spaces. For the $\pi\pi_{\mu j}$ interactions the baryons are considered to be disposed into the 4 global symmetric doublets (1a), for each of which charge independent $\pi\pi_{\mu j}$ interactions are ruled by the expression

$$L_{\pi\pi} = G \sum_{j=1}^{3} \overline{\Psi} \gamma_\mu T_j \pi\pi_{\mu j} \Psi , \tag{2}$$

$$\pi\pi_\mu = \pi\pi_{\mu 1} - i\pi\pi_{\mu 2} , \qquad \pi\pi_\mu^* = \pi\pi_{\mu 1} + i\pi\pi_{\mu 2} , \qquad \pi\pi_\mu^0 = \pi\pi_{\mu 3} , \tag{3}$$

and T_j are three 8×8 isovector matrices operating in a three dimensional isospin space

$$T_J = I^{(4)} \times \tau_j , \tag{4}$$

([6]) M. ALSTON, L. ALVAREZ, P. EBERHARD, M. GOOD, W. GRAZIANO, H. TICHO and S. WOJCICKI: *Phys. Rev. Lett.*, **6**, 300 (1961).

([7]) C. H. CHAN: *Phys. Rev. Lett.*, **6**, 383 (1961).

([8]) J. TIOMNO: *Nuovo Cimento*, **6**, 69 (1957); N. DALLAPORTA and T. TOYODA: *Nuovo Cimento*, **14**, 142 (1959); N. DALLAPORTA and V. DE SANTIS: *Nuovo Cimento*, **14**, 225 (1959); D. C. PEASLEE: *Phys. Rev.*, **117**, 873 (1960).

([9]) N. DALLAPORTA and L. K. PANDIT: *Nuovo Cimento*, **16**, 135 (1960).

([10]) N. DALLAPORTA: *Proc. of the Xth Annual International Conference on High Energy Physics* (Rochester, 1960), p. 458.

while hypercharge independent interactions are ruled by terms such as

$$L_\chi = F \sum_{\chi=1}^{4} \overline{\Psi} \gamma_\mu \omega_K \chi_{\mu K} \Psi \,, \tag{5}$$

$$\chi_\mu^+ = \chi_{\mu 1} - i\chi_{\mu 2} \,, \quad \overline{\chi_\mu} = \chi_{\mu 1} + i\chi_{\mu 2} \,, \quad \chi_\mu^0 = -\chi_{\mu 3} + i\chi_{\mu 4} \,, \quad \overline{\chi_\mu^0} = \chi_{\mu 3} + i\chi_{\mu 4} \,, \tag{6}$$

and the ω_K matrices are four 8×8 dimensional isovector matrices

$$\omega_j = -\varrho_2 \times \tau_j \times I^{(2)} \qquad (j = 1, 2, 3) \,, \qquad \omega_4 = \varrho_1 \times I^{(2)} \times I^{(2)} \,, \tag{7}$$

operating in a four-dimensional hypercharge space; in this space the four baryon doublets form a $D_{\frac{1}{2},0} + D_{0,\frac{1}{2}}$ representation and the χ_μ mesons a $D_{\frac{1}{2},\frac{1}{2}}$ representation of the rotation group. Such a scheme can be represented by the diagram of Fig. 1.

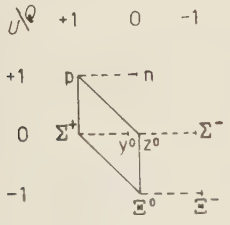

Fig. 1. – Q = electron charge, U = hypercharge, – – – π or $\pi\pi_\mu$ transitions, ——— K or χ_μ transitions.

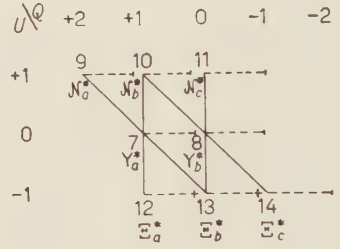

Fig. 2. – Same as Fig. 1 (the numbers of the states refer to the corresponding row in the spinor representation).

In order to avoid confusion, the K and χ_μ transitions have been indicated for only one of the pionic isospin doublets.

In all these formulae ϱ_j and τ_i are two independent sets of 2×2 Pauli matrices and $I^{(n)}$ is a unity matrix with n rows and columns; ($\times$ means direct product).

Now the real π and K pseudoscalar mesons are to be considered as complicated systems of baryons and antibaryons linked together by continuous exchanges of vector mesons (as example $\pi^+ = p + \bar{n}$, $K^0 = \Sigma^+ + p$, etc.). Such compound structures should be preferentially emitted in strong interactions in respect to the vector mesons themselves (at least at low energies) owing to their lower masses. Whatever the exact mechanism of this emission may be, as their isobaric properties are completely similar to those of the $\pi\pi_\mu$ and χ_μ vector mesons, the π_j will be emitted along the same isobaric paths in Fig. 1 as the π_j, and the K_K as the $\chi_{\mu K}$, according to interactions which will be phenomenologically described (ignoring the complicated intermediate steps) by terms such as

$$L = g \sum \overline{\Psi} \gamma_5 T_j \pi_j \Psi, \tag{8}$$

$$L_K = f \sum \overline{\Psi} \gamma_5 \omega_K K_K \Psi \,, \tag{9}$$

with similar relations as (3) and (6) defining the different charge states of the π and K mesons; g and f are now phenomenological constants which include all operations in the intermediate processes.

Now our assumption of attributing the Kπ resonance to a vector state χ_μ implies, of course, a strong interaction between χ_μ, K and pion. Owing to this interaction, it will occur that a K and a χ_μ meson successively emitted by the same baryon, will combine together to form a pion, this providing a new kind of source for pion production, whose effects from the standpoint of charge state symmetries will now be considered in some detail.

The isobaric part H_μ of the terms ruling the successive emission of a χ_μ and a K (or a K and a χ_μ)

$$L = Ff\bar{\Psi}\gamma_\mu\gamma_5\Psi H_\mu\,, \tag{10}$$

is

$$\begin{aligned} H_\mu &= \sum_{LK} \omega_L\omega_K\chi_{\mu L}K_K + \text{interchange of } \chi_\mu \text{ and K} \\ &= \sum_L \chi_{\mu L}K_L + 2i\sum_{LK} M_{LK}\chi_{\mu L}K_K + \text{interchange of } \chi_\mu \text{ and K}\,, \end{aligned} \tag{11}$$

owing to the anticommutation of the ω_K matrices, where

$$M_{LK} = -\frac{i}{4}[\omega_L\omega_K - \omega_K\omega_L]\,.$$

In what follows, we shall everywhere neglect to write explicitely the interchange terms of K and χ_μ, which should, of course, always be intended to be present.

We may further divide the antisymmetric part of (11) into a self-dual and anti-self-dual tensor term, thus obtaining

$$H_\mu = C_{\mu 0} - 2\sum_{j=1}^{3}[Y_j^+ A_{\mu j} + Y_j^- B_{\mu j}]\,, \tag{12}$$

where

$$\left\{\begin{array}{l} A_{\mu j} = \overline{\chi_\mu}U_j^+K_K \qquad B_{\mu j} = \overline{\chi_\mu}U_j^-K_K \quad \text{with} \quad K = \left|\begin{array}{c} \mathrm{K}^+ \\ \overline{\mathrm{K}}^- \\ -\,\mathrm{K}^0 \\ \overline{\mathrm{K}}^0 \end{array}\right| \overline{\chi}_\mu = |\overline{\chi_\mu^-},\ \chi_\mu^+,\ \overline{\chi_\mu^0},\ -\chi_\mu^0|\,, \\ C_{\mu 0} = \overline{\chi_\mu}K\,, \\ \text{where the } U_j^+,\ U_j^- \text{ are } 4\times 4 \text{ matrices:} \\ U_1^+ = \quad \varrho_2\times\tau_2\,, \qquad U_2^+ = -\,\varrho_2\times\tau_1\,, \qquad U_3^+ = I^{(2)}\times\tau_3\,, \\ U_1^- = -\,\varrho_2\times\tau_3\,, \qquad U_2^- = -\,\varrho_2\times I^{(2)}\,, \qquad U_3^- = \varrho_3\times\tau_3\,, \end{array}\right. \tag{13}$$

and

$$Y_j^{\pm} = \frac{M_{LK} \pm M_{j4}}{2} \qquad (LKj \text{ cyclic}),$$

$$(14) \qquad Y_j^{+} = \frac{1}{2}\left|\begin{array}{c|c} \tau_j & 0 \\ \hline 0 & 0 \end{array}\right| \times I^{(2)}, \qquad Y_j^{-} = \frac{1}{2}\left|\begin{array}{c|c} 0 & 0 \\ \hline 0 & \tau_j \end{array}\right| \times I^{(2)}.$$

The scalar term emits or absorbs $C_{\mu 0}$ doublets of strangeness and isospin zero. The second term emits or absorbs A_μ states with strangeness 2 and zero isospin equivalent to $\chi_\mu \mathrm{K}$ doublets occurring in transitions such as $\mathcal{N} \to \Xi + \mathrm{A}_\mu$; if a meson with strangeness 2 exists the χ_μ and K contained in A_μ could collapse together to form it. We have not considered further this possibility. The third term emits or absorbs B_μ states of strangeness zero and isospin 1: it is therefore, isobarically equivalent to a pion and will in fact produce it through the $\chi_\mu \mathrm{K}\pi$ interaction. It will give us a second possible source of pions which will be phenomenologically described as

$$(15) \qquad L' = g' \sum_j \Psi \gamma_5 Y_j^{-} \pi_j \Psi .$$

The g' constant includes the effects of the $\chi_\mu \mathrm{K}\pi$ interaction. According to the form of the matrices Y_j^{-} it is clear that only hyperons with hypercharge $Y_3^{+} = 0$ can be a source for such pions which will link the states Σ^+-Z^0 and Y^0-Σ^- in a second doublet arrangement. It is, therefore, clear that this second pion source will interfere with the first one (8) containing the doublets Σ^+-Y^0, Z^0-Σ^-, mixing both kinds of doublets to form the Λ singlet and Σ triplet with total isospin $I = Y^{-} + T$; $I_3 = Y_3^{-} + T_3$ as described in detail in reference (9), on purely phenomenological grounds. As a consequence, the $\Lambda\Sigma\pi$ coupling constant will turn out to be $g - g'$ and the $\Sigma\Sigma\pi$ constant $g + g'$, breaking thus partially global symmetry, while charge and hypercharge independence of π and K interactions are still preserved.

The baryonic scheme considered up to now, as in previous work, includes only the eight well-known fundamental baryonic states. However, the number of experimental resonances discovered in phenomena such as pion-nucleon scattering or pion-hyperon interactions, seem to indicate the existence of several baryonic isobars which could belong to other representations of the rotation group in the four-dimensional hypercharge space.

Too little information on these isobars and their quantum numbers is known up to now in order to try a real classification of them. What follows, therefore, should be considered as an indicative way of pointing out some possible effects to be expected from the existence of these states on the strong interaction scheme, rather than a serious attempt of extending it. In this spirit we shall just suppose the existence of isobaric states for the $D_{\frac{3}{2},1} + D_{0,\frac{1}{2}}$ representation and explore some consequences of such an assumption. The isobaric states thus introduced, are indicated by the points of diagram 2.

We shall first consider the χ_μ and K interactions acting in this scheme 2. A 16-component spinor description of Dirac's equation, corresponding to particles with spin $\frac{3}{2}$, has been given by Gupta (11):

$$(16) \qquad (\alpha_\mu \partial_\mu + m)\Psi = 0 .$$

(11) S. Gupta: *Phys. Rev.*, **95**, 1334 (1954).

The 16 components of the spinor included in it correspond to a $D_{1,\frac{1}{2}}+D_{\frac{1}{2},1}+$ $+D_{\frac{1}{2},0}+D_{0,\frac{1}{2}}$ representation; the α_μ are 16×16 matrices acting on them and playing the same role in Gupta's equation for particles of spin $\frac{3}{2}$ as the γ_μ matrices in Dirac's equation for spin $\frac{1}{2}$. We refer to Gupta's paper for their detailed expressions.

As was done in the similar case of the preceeding $D_{0,\frac{1}{2}}$ and $D_{\frac{1}{2},0}$ representations where the usual γ_μ matrices of Dirac's equation were employed as ω_K matrices to represent the rotations in four dimensional hypercharge space, we shall use here some ω_K^* matrices obtained from the α_μ matrices of Gupta to describe the K or χ_μ transitions between the different states of scheme 2; precisely, we shall choose

$$\omega_K^* = (-1)^K \alpha_K \,. \tag{17}$$

For our use, however, we shall not need all the 16 states of Gupta's equation, but only those referring to the $D_{\frac{1}{2},1}+D_{0,\frac{1}{2}}$ representations, that is, those corresponding only to the 7th till 14th lines of Gupta's operators; the $D_{0,\frac{1}{2}}$ representations refer to the states in the 7th and 8th lines, while those of the $D_{\frac{1}{2},1}$ representation to those from the 9th to the 14th line. If we now turn to the π or $\pi\pi_\mu$ interactions and still assume them as global symmetric, each of the states considered will represent a further normal isospin doublet, so that in fact, there are 16 states in all.

We shall then describe the χ_μ interactions between the isobar states, by a term

$$L_\chi^* = F \sum \Psi^* \gamma_\mu \omega_K^* \chi_{\mu K} \Psi^* \,, \tag{18}$$

where Ψ^* will be a 16-row spinor with 8 states ($Y_a^* Y_b^* \mathcal{N}_a^* \mathcal{N}_b^* \mathcal{N}_c^* \Xi_a^* \Xi_b^* \Xi_c^*$) in the rows from 7th to 14th and zeros elsewhere, each of the states being in turn a doublet.

The pion interactions will then be given by

$$L_{\pi\pi}^* = G \sum \bar{\Psi}^* \gamma_\mu T_j^* \pi\pi_{\mu j} \Psi^* \,, \tag{19}$$

with

$$T_j^* = I^{(8)} \times \tau_j \,.$$

Similar formulas to (18) and (19) with γ_5 instead of γ_μ, and K_K and π_j instead of $\chi_{\mu\pi}$ and $\pi\pi_{\mu j}$ will describe the corresponding emission of K and π mesons.

According to the previous considerations, two successive emissions of a K and a χ_μ will then be ruled by terms of type (11), where now

$$H_\mu = \sum_{LK} \omega_L^* \omega_K^* \chi_{L\mu} K_K = H_{\mu 1} + H_{\mu 2} \,, \tag{20}$$

$$H_{\mu 1} = 2i M_{LK}^* \chi_{\mu L} K_K \qquad M_{LK}^* = -\frac{i}{4} [\omega_L^* \omega_K^* - \omega_K^* \omega_L^*] \,, \tag{21}$$

$$H_{\mu 2} = 2 N_{LK}^* \chi_{\mu L} K_K \qquad N_{LK}^* = \tfrac{1}{4} [\omega_L^* \omega_K^* + \omega_K^* \omega_L^*] \,. \tag{22}$$

The antisymmetric terms are similar in all respects to those already discussed in the case of the ω_K matrices. They give rise to the following interactions:

$$H_{\mu 1} = -\tfrac{2}{3} \sum_j (- Y_j^{*+} A_j + Y^{*-} B_j) \,, \tag{23}$$

where we give in the following, only that part of the matrices referring to the 7th till 14th lines and columns:

$$
(24)\quad
\begin{cases}
Y_1^{*+} = \frac{1}{2}\begin{vmatrix} 0 & & \\ & 0 & I^{(3)} \\ & I^{(3)} & 0 \end{vmatrix} \times I^{(2)}, \qquad Y_2^{*+} = \frac{i}{2}\begin{vmatrix} 0 & & \\ & 0 & -I^{(3)} \\ & I^{(3)} & 0 \end{vmatrix} \times I^{(2)}, \\
Y_3^{*+} = \frac{1}{2}\begin{vmatrix} 0 & & \\ & I^{(3)} & 0 \\ & 0 & -I^{(3)} \end{vmatrix} \times I^{(2)}, \\
Y_j^{*-} = \frac{1}{2}\begin{vmatrix} \tau_j & & \\ & \theta_j & \\ & & \theta_j \end{vmatrix}, \qquad \theta_1 = \begin{vmatrix} & \sqrt{2} & \\ \sqrt{2} & & \sqrt{2} \\ & \sqrt{2} & \end{vmatrix}, \\
\theta_2 = i\begin{vmatrix} & -\sqrt{2} & \\ \sqrt{2} & & -\sqrt{2} \\ & \sqrt{2} & \end{vmatrix}, \qquad \theta_3 = \begin{vmatrix} 2 & & \\ & 0 & \\ & & -(-2) \end{vmatrix}.
\end{cases}
$$

Here also, the Y_j^{+*} terms clearly correspond to γ_μK meson doublets with hypercharge 2, while the Y_j^{-*} terms to γ_μK doublets of hypercharge zero (and therefore combining into a pion owing to the γ_μKπ interaction), emitted or absorbed along the horizontal lines connecting the components of the Λ^* and Ξ^* triplets and of the Y* doublet of scheme 2.

Again these pion interactions will interfere with the global symmetric ones (19) yielding:

a) for the hyperon isobars, a splitting (quite similar to the $\Lambda\Sigma$ situation) into a singlet and triplet;

b) for the nucleon isobars the splitting into a $T = \frac{3}{2}$ multiplet and a $T = \frac{1}{2}$ doublet. A similar situation holds of course, for the Ξ terms.

If we assume now that the angular momentum of the $D_{\frac{3}{2},1}$ representation is $J = \frac{3}{2}$, the terms corresponding to the $T_{\frac{3}{2}}$ multiplet would give us the well-known isobar of the $\frac{3}{2}, \frac{3}{2}$ nucleon pion resonance. Similarly, the excited hyperon triplet, would be tentatively identified with the $\Lambda\pi$ resonant state Y_1^* discovered last year [12]; its experimental J value is still uncertain; should the spin of this hyperon isobar turn out to be $\frac{3}{2}$, it would very naturally fit in the present scheme as the correspondent of the nucleon isobar in the excited representation; in the opposite case (spin $\frac{1}{2}$), the consideration of the whole $D_{\frac{3}{2},1} + D_{0,\frac{1}{2}}$ representation as a single block should be rather artificial. No experimental information concerning the possible other states of the representation, is up to now available.

Now the main difference between this isobar case and the preceeding one is that the ω_K^* matrices do not anticommute, so we shall get also a contribution of the symmetric term (22) to the transitions.

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[12] M. Alston, L. Alvarez, P. Eberhard, M. Good, W. Graziano, H. Ticho and S. Wojcicki: *Phys. Rev. Lett.*, **5**, 520 (1960).

This term can be arranged as follows:

$$H_{\mu 2} = \tfrac{1}{3} Z_{00} C_{\mu_0} - \tfrac{1}{3} \sum_{ij} Z_{ij} C_{\mu_{ij}} , \tag{25}$$

where

$$C_{\mu_{ij}} = \bar{\chi}_\mu U_i^+ U_j^- K , \tag{26}$$

$$Z_{ij} = \tfrac{1}{2} Y_i^{*+} Y_j^{*-} \times I^{(2)} ,$$

$$Z_{00} = \left| \begin{array}{c|c} 0 & \\ \hline & I^{(6)} \end{array} \right| \times I^{(2)} , \tag{27}$$

that is as a sum of a scalar and a three×three symmetric tensor. The meaning of the different components of the tensor is obvious, as the $C_{\mu_{ij}}$ represent χ_μK doublets with hypercharge ranging from zero to 2; the most interesting ones for our purpose are those containing the factor Y_3^+. In this case, owing to the χ_μKπ interaction, the two K and χ_μ of total zero hypercharge will again form a pion, yielding thus a further source which will be peculiar to the states of this $D_{\frac{1}{2},1}$ representation, that is, to the excited nucleon ($\mathcal{N}^*$) and Ξ^* state isobars only.

At this point, in order to proceed further, we can try a definite assumption concerning the K$\chi_\mu\pi$ interaction.

According to the analyis of FERRARI and coworkers [13] on K-nucleon scattering at low energies, the whole experimental situation in this field appears to be very satisfactorily explained by assuming a dominant bipion $\pi\pi_\mu$ pole contribution to the scattering, based on a strong $\bar{\mathrm{K}}$K$\pi\pi$ interaction. Moreover, in order to explain the different behaviour of K and $\bar{\mathrm{K}}$ nucleon scattering, this interaction is required to change sign with reversal of hypercharge. As our case also may be considered as a $\bar{\mathrm{K}}$K$\pi\pi$ interaction in which a K and a π combine to form the χ_μ, we should of course require the same hypercharge properties for both, and are thus led to choose a hypercharge-dependent expression. This can be easily done provided we split the two four isovectors χ_μ and K into two conjugate isospinors. We write then the χ_μKπ interaction as

$$(\bar{K}\tau_j \partial_\mu \pi_j - \partial_\mu \bar{K}\tau_j \pi_j)\chi_\mu + (K\tau_j^* \partial_\mu \pi_j - \partial_\mu K\tau_j^* \pi_j)\bar{\chi}_\mu , \tag{28}$$

where

$$K = \left| \begin{array}{c} K^+ \\ \hline -K^0 \end{array} \right| \qquad \bar{K} = \left| \begin{array}{c} \bar{K}^- \\ \hline \bar{K}^0 \end{array} \right| ,$$

and similar expressions for χ_μ and $\bar{\chi}_\mu$.

Let us now consider the transformation properties of the auxiliary pion sources we have introduced as a consequence of interaction (28). They will always be the product of a baryonic interaction term (either 11-12 or 20-25) with the χ_μKπ interaction (28). Now it may be easily checked that all the baryon terms are invariant under the transformation $\exp[i(\pi/2)Y_2^+]$ while (28) changes sign under the

[13] F. FERRARI, G. FRYE and M. PUSTERLA: *Phys. Rev. Lett.*, **4**, 615 (1960); UCRL 9421, UCRL 9434, to be published on *Phys. Rev.*; G. COSTA, F. FERRARI and M. PUSTERLA: *Nuovo Cimento*, **20**, 1224 (1961).

corresponding operation $\exp[i(\pi/2)U_2^+]$. We thus obtain in all cases hypercharge-dependent sources of pions.

It is easy to show that in the case of the (22-25) interaction terms this source of pions produces the mass splitting between Λ^0 and Ξ.

In fact, this hypercharge-dependent source pertaining to the $\mathcal{N}^*_{abc}$ and Ξ^*_{abc} states, will combine with the global symmetric source (19) giving rise to a hypercharge-dependent π interaction for the $\frac{3}{2}$ and $\frac{1}{2}$ isobars excited states $\mathcal{N}^*$ and Ξ^*. Further, we know from pion physics that there exists a strong pion interaction between the nucleon and its $\frac{3}{2},\frac{3}{2}$ isobar. According to it, a nucleon or Ξ may be thought to be virtually a kind of mixture of the $\frac{3}{2},\frac{3}{2}$ isobar and the pion. Then we shall have to consider as a further source of pion interactions for the nucleons and Ξ's, the graph (Fig. 3) which will be phenomenologically described by an interaction term of type

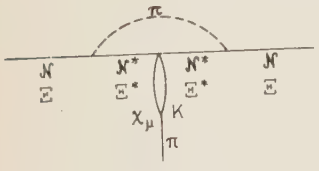

Fig. 3.

$$\Psi^* \gamma_5 (g'' + g''' Y_3^+) T_j \Psi \pi_j \,. \tag{29}$$

This will give a contribution depending on Y_3^+ which will be further summed to the global symmetric pion contribution of the $\mathcal{N}$ and Ξ interactions. Thus the coupling constants of the whole pion sources of Λ^0 and Ξ, will be different and yield different self-masses for the $\mathcal{N}$ and Ξ states.

It is to be noted that such a phenomenological explanation was already postulated in Schwinger's [14] scheme and explicitly given in formula (25) of reference [10].

Let us consider now the effect of hypercharge dependence on the hyperon pion source (15) previously considered for the $\Sigma\Lambda$ splitting. It is easy to recognize that in this case, as long as we consider $\mathcal{N}$ and Ξ to have the same mass, sign inversion for hypercharge inversion imposes that the whole source (15) of pions gives zero contribution; in fact, intermediate paths such as $\Sigma^- \to p - \bar{K}^0 \to Z^0 - \gamma_\mu^- - \overline{K^0}$; $\overline{K^0} - \overline{\gamma_\mu^-} - \pi^+$ giving a determinated contribution to the interaction are exactly compensated by the hypercharge conjugate paths such as $\Sigma^+ \to \Xi^0 + K^+ \to \Xi^0 - K^+ - \overline{\gamma_\mu^0}$; $K^- - \overline{\gamma_\mu^0} \to \pi^-$ giving the same contribution with opposite sign.

However, as soon as interaction (22-25-2?) splits the $\mathcal{N}$ and Ξ masses, then obviously the two intermediate paths will not compensate exactly anymore, and on the whole interaction (15) will not be zero, thus allowing as observed the $\Lambda\Sigma$ mass splitting. According to this view such splitting should then be in some sense a second order correction as it depends entirely on the $\mathcal{N}$-Ξ mass difference and it should perhaps reasonably be expected that its value should be smaller than the $\mathcal{N}$-Ξ one, as in fact is observed.

Thus the hypercharge-dependent character of interaction (28) and the existence of baryon isobars, allows us both a simple theoretical explanation of the baryon mass differences and the possibility of preserving the general assumption of hypercharge-independence for the whole of the other K interactions which is satisfactory as a « raison d'être » for their phenomenological aspect.

As has already been stressed, the present approach does not aim to furnish any definite general scheme for particles and excited state isobars; much too scarce experimental information is available up to now in order to have any chance of

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[14] J. SCHWINGER: *Phys. Rev.*, **104**, 1164 (1956).

making a reasonable guess for such a general problem; it is just meant to show some possible connections of different experimental aspects of the present situation which, if true, could considerably reduce the arbitrariness and number of the different assumptions necessary now to explain the phenomenology of the strong interactions.

It would turn out in fact, that three fundamental vector interactions each of which possesses high symmetries in isobaric space, (charge and hypercharge-independence and global symmetry) plus a direct γ_μK-pion interaction violating the four-dimensional rotational invariance in isospace, may account for a complete break of global symmetry and hypercharge-independence with conservation of hypercharge only; this is phenomenologically equivalent to the description of ordinary pion and K interactions through five constants g, f, g', g'' and g''' which, however, are theoretically deducible from the three fundamental ones.

Inelastic Form Factors.

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(ricevuto il 26 Luglio 1961)

We are interested by the inelastic scattering of electrons by nucleons or nuclei

$$e^- + A \to e^- + B + C ,$$

and we try to relate this process to the associated photonuclear reaction

$$\gamma + A \to B + C ,$$

by using first order calculations in the electromagnetic coupling constant

$$\alpha = \frac{e^2}{4\pi} = \frac{1}{137} .$$

1. - Let us consider the general process $e^- + A \to e^- + B + C$.

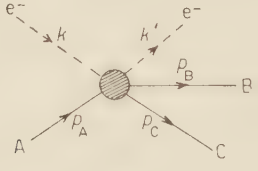

Fig. 1.

The S matrix element is given by

$$S_{fi} = \delta_{fi} - i(2\pi)^4 \delta_4(k + p_A - k' - p_B - p_C) \left(\frac{m_e^2}{k_0 k_0'}\right)^{\frac{1}{2}} \left(\frac{N_A N_B N_C}{p_A^0 p_B^0 p_C^0}\right)^{\frac{1}{2}} T_{fi} ,$$

(*) Supported in part by the United States Air Force through the European Office, Air Research and Development Command.

where m_e is the electron mass and N_i the normalization coefficients defined by

$$N_i = M_i \text{ for fermions of mass } M_i \quad N_i = \tfrac{1}{2} \text{ for bosons.}$$

The differential cross-section is given in terms of the matrix element T_{fi} by

$$(1) \qquad \mathrm{d}\sigma = \frac{1}{(2\pi)^5} \frac{m_e^2 N_A N_B N_C}{\sqrt{(k\cdot p_A)^2 - m_e^2 M_A^2}} |T_{fi}|^2 \frac{\mathrm{d}\boldsymbol{k}'}{k_0'} \frac{\mathrm{d}\boldsymbol{p}_B}{p_B^0} \frac{\mathrm{d}\boldsymbol{p}_C}{p_C^0} \delta_4(k + p_A - p_B - p_C - k') .$$

2. – We use the first Born approximation for the electromagnetic interaction and we replace the general diagram of Fig. 1 by the following one:

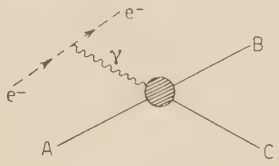

Fig. 2.

The T matrix is the product of two matrix elements for the electromagnetic current j_μ:

$$(2) \qquad T_{fi} = \frac{1}{q^2} \langle k' | j_\mu | k \rangle \langle BC | j_\mu | A \rangle .$$

The first one is simply

$$\langle k' | j_\mu | k \rangle = e\, \overline{u}_{s'}(k') \gamma_\mu u_s(k) ,$$

where $u_s(k)$ is a free Dirac spinor describing an electron of energy momentum k and spin s.

The second one describes the photoreaction $\gamma + \mathrm{A} \to \mathrm{B} + \mathrm{C}$ for a virtual photon and is unknown.

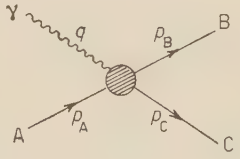

Fig. 3.

3. – We have now to calculate $|T_{fi}|^2$ with the factorized form (2):

$$(3) \qquad |T_{fi}|^2 = \frac{1}{q^4} \left[\frac{1}{2} \sum_{s'} \langle k' | j_\mu | k \rangle \langle k' | j_\nu | k \rangle^* \right] \left[\frac{1}{2s_A + 1} \sum_{s_A s_B s_C} \langle BC | j_\mu | C \rangle \langle BC | j_\nu | C \rangle^* \right] .$$

The summation over electron spins is well known and we obtain

$$|T_{fi}|^2 = \frac{1}{q^4}\frac{e^2}{2m_e^2}[k_\mu k'_\nu + k_\nu k'_\mu + \tfrac{1}{2}q^2\delta_{\mu\nu}]T_{\mu\nu}\,. \tag{4}$$

The tensor $T_{\mu\nu}$, only related to the reaction $\gamma + A \to B + C$, must be constructed with the energy-momentum four-vectors q, p_A, p_B, p_C. Because of energy-momentum conservation we can use only q, p_A, p_B. Contributions proportional to q vanish after saturation with the electron part $(q = k - k')$ which is a symmetrical tensor in μ and ν. The general form of $T_{\mu\nu}$ can then be written as

$$T_{\mu\nu} = \delta_{\mu\nu}\mathscr{T}_0 + p_{A\mu}p_{A\nu}\mathscr{T}_1 + p_{B\mu}p_{B\nu}\mathscr{T}_2 + \tfrac{1}{2}(p_{A\mu}p_{B\nu} + p_{A\nu}p_{B\mu})\mathscr{T}_3\,. \tag{5}$$

The functions $\mathscr{T}_i$ depend only on three scalar variables; it is convenient to choose q^2, W^2 and $Z = \cos\theta$ where θ is the angle between $\boldsymbol{p}_A$ and $\boldsymbol{p}_B$ in the c.m. system for particles B and C. Eq. (4) can be written in the form:

$$|T_{fi}|^2 = \frac{e^2}{2m_e^2}\frac{1}{q^4}\sum_i T_i\,\mathscr{T}_i(q^2, W^2, Z)\,, \tag{6}$$

where

$$\begin{cases} T_0 = q^2 - 2m_e^2\,, \\ T_1 = 2(k\cdot p_A)(k'\cdot p_A) - \tfrac{1}{2}q^2M_A^2\,, \\ T_2 = 2(k\cdot p_B)(k'\cdot p_B) - \tfrac{1}{2}q^2M_B^2\,, \\ T_3 = (k\cdot p_A)(k'\cdot p_B) + (k\cdot p_B)(k'\cdot p_A) + \tfrac{1}{2}q^2(p_A\cdot p_B)\,. \end{cases} \tag{7}$$

4. – In the c.m. of B and C, eq. (1) can be transformed into

$$\mathrm{d}\sigma = \frac{e^2}{2(2\pi)^5}\frac{N_A N_B N_e}{\sqrt{(k\cdot p_A)^2 - m_e^2M_A^2}}\frac{1}{q^4}\sum_i T_i\,\mathscr{T}_i(q^2, W^2, Z)\,\frac{\mathrm{d}_3\boldsymbol{k}'}{k'_0}\,\frac{K_B}{W}\,\mathrm{d}\Phi\,\mathrm{d}Z\,, \tag{8}$$

where Φ is the angle of the ABC plane with respect to the electron plane.

The Φ dependence is explicitly given in the T_i functions; the integration over Φ can be easily performed:

$$2\pi\Pi_i = \int_0^{2\pi} T_i\,\mathrm{d}\Phi\,,$$

and the result is the following:

$$\begin{cases} \Pi_0 = T_0\,, \\ \Pi_1 = T_1\,, \\ \Pi_2 = \dfrac{1}{2}(1-Z^2)K_B^2T_0 + \dfrac{1}{W^2}\left[\left(E_B + \dfrac{K_B}{K_A}q_0Z\right)^2 + \dfrac{q^2}{2}\dfrac{K_B^2}{K_A^2}(1-Z)^2\right]T_1\,, \\ \Pi_3 = \dfrac{1}{W}\left(E_B + \dfrac{K_A}{K_B}q_0Z\right)T_1\,. \end{cases} \tag{9}$$

By putting

$$U_0(q^2, W^2, Z) = \mathscr{T}_0 + \tfrac{1}{2}1(-Z^2)K_B^2\mathscr{T}_2 ,$$

$$U_1(q^2, W^2, Z) = \mathscr{T}_1 + \frac{1}{W^2}\left[\left(E_B + \frac{K_B}{K_A}q_0 Z\right)^2 + \frac{q^2}{2}\frac{K_B^2}{K_A^2}(1-Z^2)\right]\mathscr{T}_2 + \frac{1}{W}\left(E_B + \frac{K_A}{K_B}q_0 Z\right)\mathscr{T}_3,$$

the differential cross section is given by

$$(10) \qquad d\sigma = \frac{e^2}{4\pi}\frac{1}{(2\pi)^3}\frac{N_A N_B N_e}{\sqrt{(k\cdot p_A)^2 - m_e^2 M_A^2}}\frac{1}{q^4}\frac{K_B}{W}[T_0 U_0 + T_1 U_1]\frac{d_3\boldsymbol{k}'}{k_0'}\,dZ .$$

5. – We are interested by experiments where only the final electron is observed and we must integrate over the B and C variables. By defining

$$V_{0,1}(q^2, W^2) = \int_{-1}^{+1} U_{0,1}(q^2, W^2, Z)\,dZ ,$$

we transform (10) into

$$(11) \qquad d\sigma = \frac{\alpha}{(2\pi)^3}\frac{N_A N_B N_C}{\sqrt{(k\cdot p_A)^2 - m_e^2 M_A^2}}\frac{1}{q^4}\frac{K_B}{W}[T_0 V_0 + T_1 V_1]\frac{d_3\boldsymbol{k}'}{k_0'} .$$

It is convenient to use laboratory variables for the electrons:

$$(12) \qquad \frac{\partial^2\sigma}{\partial k_0'\,\partial\cos\theta} = \frac{\alpha}{(2\pi)^2}\frac{N_A N_B N_C}{M_A}\frac{k'}{k}\frac{1}{q^4}\frac{K_B}{W}[T_0 V_0 + T_1 V_1] .$$

In the bracket of eq. (12), only T_1 is function of the incident energy k_0 independently of q^2 and W^2. We define two inelastic form factors $V_0(q^2, W^2)$ and $V_1(q^2, W^2)$ uniquely related to the photoreaction $\gamma + A \to B + C$ for a virtual photon. By performing two experiments at different energies k_0 and angles θ choosen to give the same q^2 and W^2 values, we can reach the two inelastic form factors $V_0(q^2, W^2)$ and $V_1(q^2, W^2)$.

6. – It is possible to relate V_0 and V_1 to the polarized cross-sections for the process $\gamma + A \to B + C$.

By using the same c.m. variables as previously, the differential cross-section is given by

$$(13) \qquad \frac{d\sigma_P}{dZ} = \frac{1}{4\pi}\frac{N_A N_B N_C}{W^2}\frac{K_B}{K_A}\sum_{\text{pol.}} e_\mu e_\nu T_{\mu\nu} .$$

Summation over the photon polarization can be performed by using

$$\sum_{\text{pol}} e_\mu e_\nu = \delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} .$$

After straightforward calculations, one can split the differential cross-section into transverse and longitudinal parts

$$\frac{\mathrm{d}\sigma_P}{\mathrm{d}Z} = 2\,\frac{\mathrm{d}\sigma_T}{\mathrm{d}Z} + \frac{\mathrm{d}\sigma_L}{\mathrm{d}Z},$$

with

$$(14)\qquad \begin{cases} \dfrac{\mathrm{d}\sigma_T}{\mathrm{d}Z} = \dfrac{1}{4\pi}\,\dfrac{N_A N_B N_C}{W^2}\,\dfrac{K_B}{K_A}\,U_0(q^2, W^2, Z)\,, \\ \dfrac{\mathrm{d}\sigma_L}{\mathrm{d}Z} = \dfrac{1}{4\pi}\,\dfrac{N_A N_B N_C}{W^2}\,\dfrac{K_B}{K_A}\left[U_0(q^2, W^2, Z) - \dfrac{K_A^2 W^2}{q^2}\,U_1(q^2, W^2, Z)\right]. \end{cases}$$

The inelastic cross-section (12) can be expressed in terms of σ_T and σ_L

$$(15)\qquad \frac{\partial^2\sigma}{\partial k_0'\,\partial\cos\theta} = \frac{\alpha}{\pi}\,\frac{k'}{k}\,\frac{1}{M_A}\,\frac{K_A W}{q^4}\left\{\sigma_T(q^2, W^2)\,T_0 + [\sigma_T(q^2, W^2) + \sigma_L(q^2, W^2)]\,\frac{q^2}{K_A^2 W^2}\,T_1\right\}.$$

7. – For relativistic electrons, ($k_0 \simeq k$; $k_0' \simeq k'$) the functions T_0 and T_1 can be approximated by

$$T_0 = q^2 = 4kk'\sin^2\frac{\theta}{2}, \qquad T_1 = 2kk'\cos^2\frac{\theta}{2} = \frac{q^2}{2}\,\mathrm{ctg}^2\frac{\theta}{2}.$$

The inelastic cross-section (12) takes then a form analogous to the Rosenbluth formula:

$$(16)\qquad \frac{\partial^2\sigma}{\partial k_0'\,\partial\cos\theta} = \frac{\alpha}{8\pi^2}\,\frac{N_A N_B N_C}{M_A}\,\frac{\cos(\theta/2)}{4k^2\sin^4(\theta/2)}\left[V_1(q^2, W^2) + 2\,\mathrm{tg}^2\frac{\theta}{2}\,V_0(q^2, W^2)\,\frac{K_B}{W}\right].$$

Expression (16) shows that the best experiments to reach V_0 and V_1 are backward-forward experiments. But the possibility to perform such experiments for all interesting values of q^2 and W^2 depends strongly on the kinematics. It can be easily understood that backward-forward experiments require much different incident energies because of the relation $q^2 \simeq 4kk'\sin^2(\theta/2)$; this situation is analogous to the corresponding one for elastic form factors.

The present investigation can be applied, in particular to electrodisintegration of deuterium and electroproduction of pions on nucleons. In the latter case, it will be convenient to study the form factors V_0 and V_1 in the neighbourhood of the $\frac{3}{2}\,\frac{3}{2}$ π-$\mathcal{N}$ resonance. This allows to construct simple models where the relation between $V_0(q^2, W^2)$, $V_1(q^2, W^2)$ and the nucleon electromagnetic form factors is dominated by the photoproduction amplitudes (extrapolated for $q^2 \neq 0$) corresponding to this resonance.

8. – This investigation was carried out during a stage of the author in the « Istituto di Fisica dell'Università di Bologna »; he wishes to express his thanks to Professor PUPPI for the very kind hospitality extended to him and to acknowledge Professors FUBINI, STANGHELLINI and MARTIN for illuminating discussions on this subject.

Note added in proof.

The existence of two form factors for the differential cross section is a well-known and unpublished result of many authors. But it is very important from a practical point of view to know the physical interpretation and the normalization for zero momentum transfer for these form factors.

Resonance in the K-π System.

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(ricevuto il 28 Agosto 1961)

The p wave resonance in the pion-pion scattering has been predicted by Takeda *et al.* [1] in order to explain the 2nd and 3rd maximum of the π-N scattering cross section, and by Fulco and Frazer [2] for the explanation of the isovector form factor of the anomalous magnetic moment. The theoretical possibility of pion-pion resonance was investigated by one of the authors [3] and by Chew and Mandelstam [4]. The p wave K-π resonance can also be anticipated by the analysis of one of the authors. Recently Alston *et al.* [5] have discovered the unstable K* particle in the $K^- + p \to \bar{K}^0 + \pi^- + p$ reaction, and its mass m^* is 885 MeV and half width is 8 MeV. In this note we shall analyse the possibility of the identification of the K* particle as K-π resonance by the previous method used for the explanation of pion-pion resonance.

The p wave K-π scattering phase shift can be evaluated by the Chew-Mandelstam method, if the scattering length « a » is given.

The phase shift is given as follows.

$$(1) \qquad (2\nu^{\frac{3}{2}}/(\sqrt{1+\nu}+\sqrt{m^2+\nu}))\operatorname{ctg}\delta = \frac{1}{\alpha}\left\{1 - F(\nu)\right\},$$

where ν - square of the three-momentum, $\alpha = [(1+m)/2]a$, m - mass of the K-meson.

$$(2) \quad F(\nu) = \frac{\alpha\nu}{\pi}\int_0^A \frac{\mathrm{d}\nu' 2\nu'^{\frac{3}{2}}}{\nu'(\nu'-\nu)(\sqrt{1+\nu}+\sqrt{m^2+\nu})} - \frac{2\alpha\nu}{\pi}\left(\log 2A + \frac{1}{2} + \frac{f(m^2)-f(1)}{m^2-1}\right),$$

where

$$f(\nu) = -2\sqrt{\nu(m^2+\nu)}\ln\frac{\sqrt{m^2+\nu}+\sqrt{\nu}}{m} - \left(\nu+\frac{1}{2}m^2\right)\ln m^2 .$$

[1] G. Takeda, M. Kato, K. Itabashi and K. Nakagawa: *Progr. Theor. Phys.*, **24**, 529 (1960).
[2] J. R. Fulco and W. R. Frazer. *Phys. Rev.*, **117**, 1609 (1960).
[3] Y. Miyamoto: *Progr. Theor. Phys.*, **24**, 840 (1960).
[4] G. F. Chew and S. Mandelstam: *Nuovo Cimento*, **19**, 752 (1961).
[5] M. Alston, L. W. Alvarez, P. Eberhard and M. L. Good: *Phys. Rev. Lett.*, **6**, 300 (1961).

We use the unit (pion mass = 1). A is a cut-off of the dispersion integral. We interpret this cut-off as follows. If we take into account properly the effect of nucleon and antinucleon and their virtual annihilation, the virtual integral begins to drop at $\nu = 2M$, and we can get a convergent result without any cut-off factor. Hence we take $A = 2M$ (M nucleon mass). This is confirmed by YAMAZAKI ([6]) in the Tamm-Dancoff method of π-π scattering. From $F(\nu_r) = 1$ and $\sqrt{1+\nu_r} + \sqrt{m^2+\nu_r} = 885$ MeV, we can determine scattering length $a = 0.09(\hbar/\mu c)$ ([3]). (1) can be rewritten in one level resonance formula

$$\operatorname{ctg}\delta = \frac{\sqrt{1+\nu_r} - \sqrt{1+\nu}}{\Gamma/2}, \tag{3}$$

where

$$\Gamma/2 = \frac{\alpha \nu_r^{\frac{3}{2}}}{m^* F'(\nu_r)\sqrt{1+\nu_r}} = 100 \text{ MeV}.$$

The level width seems to be considerably larger than the experimental one, though the statistics are not so good at present. If the experimental narrow width is correct, our model must be abandoned. The K* particles must be regarded as a compound system of some heavier particles (bound system of Λ and $\mathcal{N}$ in the Sakata model) ([7]), or the bound state of one K and several π-mesons). This narrow width may be due to a selection rule which operates between the K* and K + π systems. One example of selection is a generalized G-parity $U = GR$ ([8]). G is a conventional G parity and R is an operator of exchange of $\mathcal{N}$ and Ξ. We take the following Lagrangian.

$$L = ig\tilde{\boldsymbol{\psi}}(\tau\varphi)\boldsymbol{\psi} + if\tilde{\boldsymbol{\psi}}\gamma_5(\tau\Sigma + \Lambda)K + if'\tilde{\boldsymbol{\psi}}\gamma_\mu(\tau\Sigma + \Lambda)\zeta_3\boldsymbol{K}^* + \text{comp. conj.}, \tag{4}$$

where

$$\boldsymbol{\psi} = \begin{pmatrix} N \\ \Xi \end{pmatrix}, \qquad \boldsymbol{K} = \begin{pmatrix} K \\ -i\tau_2\bar{K}^{tr} \end{pmatrix}, \qquad \zeta_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and φ is a pion field. Our Lagrangian is invariant under the U operation

$$\boldsymbol{\psi} \to Ci\tau_2\zeta_2\tilde{\boldsymbol{\psi}}, \qquad \tilde{\boldsymbol{\psi}} \to -\boldsymbol{\psi}C^{-1}(i\tau_2 i\zeta_2), \qquad \Sigma \to C\bar{\Sigma}, \qquad \bar{\Sigma} \to -\Sigma C^{-1},$$

$$\Lambda \to -C\bar{\Lambda}, \qquad \bar{\Lambda} \to \Lambda C^{-1}, \qquad \boldsymbol{K} \to -i\tau_2 i\zeta_2\bar{\boldsymbol{K}}^{tr},$$

$$\boldsymbol{K}_\mu^* \to -i\tau_2 i\zeta_2\bar{\boldsymbol{K}}^{*tr}, \qquad \varphi \to -\varphi.$$

([6]) M. YAMAZAKI: *Progr. Theor. Phys.*, **25**, 727 (1960).

([7]) S. SAKATA: *Progr. Theor. Phys.*, **16**, 686 (1956).

([8]) H. FUKUDA and Y. MIYAMOTO: *Progr. Theor. Phys.*, **4**, 389 (1950); M. KAWAGUCHI and K. NISHIJIMA: *Phys. Rev.*, **108**, 905 (1957); T. D. LEE and C. N. YANG: *Nuovo Cimento*, **3**, 749 (1956).

With the help of the identity $\boldsymbol{K}=i\tau_2 i\zeta_2\overline{\boldsymbol{K}}^{tr}$, bosons simply transform as $\boldsymbol{K}\to-\boldsymbol{K}$, $\boldsymbol{K}^*_\mu\to-\boldsymbol{K}^*_\mu$ and $\varphi\to-\varphi$. Hence we can define U parity. By this theorem decay of K*-meson into K and π-mesons is forbidden, since the product of U parities of K*, K and π-mesons is odd. We introduce the ζ_3-matrix in (4) to give odd U-parity to K*. By slight violation of the Λ-Ξ symmetry, the K*-meson will decay slowly into K and π-mesons. Without ζ_3-matrix in (4) (*) the decay of the K*-meson of spin 0 is forbidden. If the K*-meson is considered as a bound state of K and an even number of pions (9), $K^*\to K+\pi$ is also forbidden owing to U-parity conservation.

* * *

The authors would like to express their sincere gratitude to Professor N. FUKUDA for his encouragement in this work.

(9) G. F. CHEW: *Phys. Rev. Lett.*, **4**, 142 (1960).

(*) We generalize the theorem of ref. (9) as follows: transitions involving π, K and K* mesons are forbidden if the number of π and K mesons $+n(v)+n(t)+n(\zeta_3)=$ odd, where $n(v)$, $n(t)$, and $n(\zeta_3)$ are numbers of vector and tensor couplings and ζ_3, the spin of the K* meson respectively.

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Parametric Dispersion Relations for Electron Feynman Amplitudes.

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(ricevuto il 4 Settembre 1961)

By extending the manifestly covariant form of the Feynman notation to the mass coordinate and its Fourier transform, the proper time coordinate ([1]), we should like to derive in an alternate way and explicitly the Muraskin-Nishijima parametric dispersion relations ([2]) for the particular case of Dirac-Maxwell equations. Thereby an interpretation of the subtraction terms in the general field theory proposed by them is also obtained.

Let $K_\varkappa(12)$ and $L(12)$ be the expectation values of the free electron and photon Feynman propagation operators ([3]). It is well known ([4]) that if

$$X(12) = \frac{1}{(2\pi)^4}\int \exp[ip\cdot(12)]\,X(p)\,\mathrm{d}^4p\,, \tag{1}$$

then for $\varepsilon \to +0$

$$K_\varkappa(p) = \frac{i}{i\gamma\cdot p + \varkappa - i\varepsilon}\,, \qquad L(p) = \frac{i\hbar c}{p^2 - i\varepsilon}\,, \tag{2}$$

$$K_\tau(p) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\,[i\varkappa\tau]K_\varkappa(p)\,\mathrm{d}\varkappa = 0 \qquad \text{for } \tau < 0, \tag{3}$$

$$\left\{\begin{aligned} &\frac{\mathscr{P}}{i\gamma\cdot p + \varkappa} = -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\mathscr{P}}{\varkappa' - \varkappa}\,\pi\,\delta(i\gamma\cdot p + \varkappa')\,\mathrm{d}\varkappa' = \mathscr{I}\,K_\varkappa(p)\,,\\ &\pi\,\delta(i\gamma\cdot p + \varkappa) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\mathscr{P}}{\varkappa' - \varkappa}\,\frac{\mathscr{P}}{i\gamma\cdot p + \varkappa'}\,\mathrm{d}\varkappa' = \mathscr{R}\,K_\varkappa(p)\,, \end{aligned}\right. \tag{4}$$

([1]) J. Schwinger: *Phys. Rev.*, **82**, 664 (1951).

([2]) M. Muraskin and K. Nishijima: *Phys. Rev.*, **112**, 331 (1961).

([3]) The notation of P. Sen: *Nuovo Cimento*, **13**, 1122 (1959) is used here.

([4]) N. N. Bogoliubov and D. V. Shirkov: *Introduction to the Theory of Quantized Fields*. Appendix 1 (New York, 1959).

where $\mathscr{P}$, $\mathscr{I}$, $\mathscr{R}$ denote the principle, imaginary and real parts,

$$\varkappa = mc/\hbar \qquad \text{and} \qquad \delta(i\dot{\gamma}\cdot p + \varkappa) = (-i\dot{\gamma}\cdot p + \varkappa)\,\delta(p^2 + \varkappa^2)\,.$$

Thus $K_\varkappa(p)$ is an analytic function in the complex $\varkappa$ plane. $\mathscr{P}_\varkappa(i\dot{\gamma}\cdot p - \varkappa)$ and $\pi\,\delta(ik\cdot p + \varkappa)$ are conjugate Hilbert transforms [5] and it is sufficient that either of the relations (3) or (4) hold for the validity of the other. Here we have also extended the Feynman notation to τ and $\varkappa$ coordinates,

$$K_\tau(x) = \frac{1}{(2\pi)^5}\iint \exp\,[i(p\cdot x + \varkappa\tau)]K_\varkappa(p)\,\mathrm{d}^4p\,\mathrm{d}\varkappa\,. \tag{5}$$

Let the general Feynman electron propagation operator be denoted by $K'_\varkappa(1, 2)$. In momentum representation its amplitude $\langle k'_i\,|\,K'_\varkappa(p', p'')\,|\,k''_f\rangle$ of η initial and ζ final photons of energy momentum k'_i and k''_f respectively, so that $p'' = p' + \sum_{i=1}^{\eta} k'_i - \sum_{f=1}^{\zeta} k''_f$, is

$$\langle k'_i\,|\,K'_\varkappa(p', p'')\,|\,k''_f\rangle = \sum_{n=0}^{\infty}\sum_{\substack{\text{all graphs of}\\ \text{order } n}} \lambda^{2n+\eta+\zeta}\int\!\ldots\!\int K_\varkappa(p')\cdot \tag{6}$$

$$\cdot F^{(n)}\{K_\varkappa(p', k'_i, k''_f, k_1 \ldots k_n), L(p', k'_i, k''_f, k_1 \ldots k_n), A^e(k'_i, k''_f)\}K_\varkappa(p'')\,\mathrm{d}^4k_1 \ldots \mathrm{d}^4k_n\,,$$

where $\lambda = e/\hbar c$, A^e is the external potential, $F^{(n)}\{K_\varkappa, L, A^e\}$ is a product of

$$(2n - 1 + \eta + \zeta)\,K_\varkappa\text{'s}\,, \qquad n\,L\text{'s} \qquad \text{and} \qquad (n + \zeta)\,A^e\text{'s}\,,$$

and the particular Feynman graph determines how each of the $K_\varkappa$'s and L's depend on p', k'_i and k''_f.

Then by means of (3) it is noted that

$$\int_{-\infty}^{\infty}\exp\,[i\varkappa\tau]K_\varkappa(p')F^{(n)}\{K_\varkappa, L, A^e\}K_\varkappa(p'')\,\mathrm{d}x = \tag{7}$$

$$= \int_{-\infty}^{\infty}\!\ldots\!\int_{-\infty}^{\infty}\exp\,[i\varkappa(\tau - \tau_1 - \ldots - \tau_{2n+1+\eta+\zeta})]K_{\tau_1}(p')\cdot$$

$$\cdot F^{(n)}\{K_{\tau_m}, L, A^e\}K_{\tau_{2n+1+\eta+\zeta}}(p'')\,\mathrm{d}\varkappa\,\mathrm{d}\tau_1 \ldots \mathrm{d}\tau_{2n+1+\eta+\zeta} =$$

$$= \int_{-\infty}^{\infty}\!\ldots\!\int_{-\infty}^{\infty}\delta(\tau - \tau_1 - \ldots - \tau_{2n+1+\eta+\zeta})K_{\tau_1}(p')\cdot$$

$$\cdot F^{(n)}\{K_{\tau_m}, L, A^e\}K_{\tau_{2n+1+\eta+\zeta}}(p'')\,\mathrm{d}\tau_1 \ldots \mathrm{d}\tau_{2n+1+\eta+\zeta} = 0\,, \qquad \text{for } \tau < 0,$$

[5] E. C. Titchmarsh: *Theory of Fourier Integrals*, Second Edition, Theorem 95 (Oxford, 1948).

where $F^{(n)}\{K_{\tau_m}, L, A^e\}$ is obtained from $F^{(n)}\{K_\varkappa, L, A^e\}$ by replacing its first $K_\varkappa$ by K_{τ_2}, ..., and the final $K_\varkappa$ by $K_{\tau_{2n+1+\eta+\zeta}}$. Therefore we obtain an alternate proof of Källén-Lehmann's integral representation ([2]) as well as the parametric dispersion relations

$$\mathscr{I}\langle k_i' | K_\varkappa'(p', p'') | k_f''\rangle = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\mathscr{P}}{\varkappa' - \varkappa}\, \mathscr{R}\langle k_i' | K_{\varkappa'}'(p', p'') | k_f''\rangle \, \mathrm{d}\varkappa', \tag{8}$$

$$\mathscr{R}\langle k_i' | K_\varkappa'(p', p'') | k_f''\rangle = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\mathscr{P}}{\varkappa' - \varkappa}\, \mathscr{I}\langle k_i' | K_{\varkappa'}'(p', p'') | k_f''\rangle \, \mathrm{d}\varkappa', \tag{9}$$

which can then he correlated to the physical scattering amplitudes ([2]) and perhaps another mode of utilizing them is by transforming them to the energy coordinates by the relation $E^2 = \boldsymbol{p}^2 + \varkappa^2$; $E\,\mathrm{d}E = \varkappa\,\mathrm{d}\varkappa$.

We note the absence of subtraction terms in the dispersion relations (8) and (9) which suggests the interpretation that the Muraskin-Nishijima subtraction terms ([2]) are equivalent to Dyson's renormalization factors ([6]) and that the Muraskin-Nishijima and Dyson field theories are equivalent. However if instead of the Feynman amplitudes (6) the causal Feynman amplitudes ([7]) defined to be the expectation values of the causal Feynman propagator ([3])

$$\begin{cases} K_{c,\varkappa}'(1, 2) = \begin{cases} \frac{1}{2}\{\overline{\psi}_\varkappa'(1), \psi_\varkappa'(2)\}\,, & \text{for } x_0(1) > x_0(2), \\ -\frac{1}{2}\{\overline{\psi}_\varkappa'(1), \psi_\varkappa'(2)\}\,, & \text{for } x_0(1) < x_0(2), \end{cases} \\ \left(\gamma\cdot\dfrac{\partial}{\partial 1} - i\lambda(a'(1) + a^e(1)) + \varkappa\right) K_{c,\varkappa}'(1, 2) = i\delta(12)\,, \\ K_{c,\varkappa}(12) = K_{c,\varkappa}'(1, 2) \quad \text{for } \lambda = 0\,, \quad K_{c,\varkappa}(p) = \mathscr{P}/(i\gamma\cdot p + \varkappa)\,, \end{cases} \tag{10}$$

are evaluated then equations similar to those of the ref. ([3]) are obtained and it is possible that renormalization can be eliminated from the Dirac-Maxwell equations in accordance with Muraskin and Nishijima.

([6]) F. J. Dyson: *Phys. Rev.*, **75**, 486, 1736 (1949).
([7]) Y. Nanbu: *Phys. Rev.*, **98**, 803 (1955); **100**, 394 (1955).

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Recensioni.

T. E. STERNE – *An Introduction to Celestial Mechanics*. Interscience Tracts on Physics and Astronomy, no. 9. Interscience Publishers Inc., New York, N.Y., pp. XI-206; cloth: $ 4.50; paper: $ 2.50.

Col lancio dei satelliti artificiali terrestri la Meccanica Celeste va riprendendo, dopo una parentesi di circa 30 anni, quel ruolo di primaria importanza che aveva assunto nello scorso secolo e nel primo ventennio del secolo presente, raggiungendo il suo apice con l'opera del Poincaré, del Sunderman, del Levi-Civita.

Appartiene a quest'epoca, che possiamo chiamare aurea, il ben noto trattato del Moulton, magistrale introduzione alla lettura dei più poderosi trattati del Tisserand e dello Charlier, e alle geniali pagine delle *Methodes Nouvelles* del Poincaré.

Con tali esempi davanti era vera-

mente difficile riproporre una introduzione alla Meccanica Celeste che uscisse fuori dai limiti di una attenta compilazione. Per riuscirvi bisognava porre l'accento sul fatto nuovo: il satellite artificiale, e questa esigenza è stata ben compresa dall'autore del libro che stiamo esaminando. Egli ha perciò il merito di avere scritto un'opera veramente utile, che inquadra nel campo più vasto della Meccanica Celeste il problema attuale del satellite artificiale.

Il volume si divide in 6 capitoli, di questi il terzo (orbite nello spazio, effemeridi, precessione e nutazione) e il quarto (variazione degli elementi ellittici, perturbazioni generali) possono considerarsi un compendio di teorie classiche. Invece gli altri capitoli, pur non contenendo contributi strettamente originali, si distinguono per il continuo riferimento al satellite artificiale. In particolare, si noterà nel Capitolo primo (problema dei due corpi) la sistematica impostazione del problema del moto a partire dalla posizione e velocità iniziale e la scelta delle unità di misura (raggio equatoriale terrestre come unità di lunghezza, e una nuova unità di tempo, che rende unitario il prodotto della costante attrattiva per la massa terrestre), scelta questa, che facilita assai i calcoli di massima della traiettoria dei satelliti. Anche in certo senso nuovo può dirsi il Capitolo secondo, per la succinta impostazione del problema geodetico, che pone in relazione i termini dello sviluppo in serie di funzioni sferiche del potenziale terrestre con le anomalie del Geoide. Forse avremmo gradito un riferimento concettuale ai fondamentali teoremi geodetici di Stokes, e ai fondamentali contributi, in questo campo, del Pizzetti e del Somigliana. Infine i due ulteriori capitoli trattano delle perturbazioni dei satelliti o dell'integrazione numerica delle equazioni del moto.

Molto ben scelti gli esempi numerici e gli esercizi, assai utili per la messa a punto di problemi particolari.

Complessivamente si tratta di una opera di grande utilità per coloro che desiderano affrontare problemi di calcolo di orbite di satelliti artificiali, specialmente a scopo geodetico. Manca completamente (e volutamente) ogni accenno a questioni critiche (integrali primi uniformi, regolarizzazione del problema dei tre corpi, orbite periodiche, asintotiche, ecc.).

M. CIMINO

R. RESNICK and D. HOLLIDAY: *Physics for Students of Science and Engineering*. Part I - John Wiley Co., 1960, pp. XIV+574; $ 6.00.

Si tratta di un corso di fisica per studenti del primo biennio, in due volumi, in molti aspetti simile a quello che normalmente svolgiamo in Italia nel corso di Fisica Sperimentale. A prima vista si rimane favorevolmente impressionati dalla bellezza delle figure e dall'articolazione dei caratteri, ma un esame appena più ravvicinato mostra quanto di reclamistico e fumettistico ci sia dietro questa presentazione. Così a p. 9, nel primo capitolo sulle misure di fisica, l'intera pagina è dedicata ad illustrare l'orologio molecolare basato sulle vibrazioni dell'ammoniaca, e descritto in termini di curve d'assorbimento, risonanza, ecc.

Per quanto crediamo sinceramente che questo modo di fare sia stato dettato dal desiderio di attrarre i ragazzi americani alla fisica, non ci sembra che questo sia il modo di procedere. Non si tratta di eccitare i giovani con quello che non possono capire, ma di far loro capire solo qualche cosa spiegandola per bene.

Saltando alla fine del primo volume, si rimane ancora impressionati dal fatto che tutta la trattazione dell'entropia sia contenuta in sei pagine (figure ed esempi inclusi), ma che si trovi anche il modo

di citare le temperature negative. Incidentalmente notiamo pure che l'entropia viene presentata come « capacità del sistema di fare del lavoro », il che può essere presentato come un punto di arrivo e non di partenza. Forse la parte migliore del libro è costituita dalla meccanica (specialmente per il capitolo sulle collisioni e quello sulle vibrazioni) dove la matematica è assai ridotta, e la fisica è svolta con ricchezza di esempi e figure. Infine ogni capitolo termina con una lunga serie di « questions » spesso molto ben poste e che secondo noi sono la cosa più interessante del libro.

G. Careri

P. Rouard – *Electroacoustique*. Ed. Libraire Armand Colin, Paris, 1960.

L'analogia messa in evidenza da P. Curie nel 1891 fra l'equazione del moto di una massa puntiforme soggetta a forza di richiamo e forza d'attrito e l'equazione del moto di cariche elettriche in un circuito comprendente capacità, induttanza e resistenza, ha permesso di stabilire il concetto di equivalenza formale fra grandezze elettriche da una parte e grandezze meccaniche ed acustiche dall'altra.

Si possono così utilizzare nella meccanica e nell'acustica tutte le conoscenze sul comportamento dei circuiti elettrici in modo che lo studio dei sistemi meccanici per la generazione, la ricezione e la registrazione del suono possa sempre ricondursi allo studio di particolari « circuiti equivalenti ».

Nello stabilire queste analogie ci si limita, in generale, al caso semplice in cui i sistemi in esame siano rappresentabili con equazioni differenziali lineari, ritenendo peraltro che la soluzione sia corretta, in prima approssimazione, anche nel caso non lineare, purchè si considerino movimenti di piccola ampiezza.

Si è così sviluppata, a fianco dell'acustica classica, un'acustica moderna — essenzialmente fondata sui concetti dell'analogia — forse meno rigorosa, ma non per questo meno intuitiva ed efficace nella soluzione dei problemi pratici posti dalle tecniche elettroacustiche, oggi in avanzata evoluzione.

« Electroacustique » di Pierre Rouard — volumetto di piccolo formato di circa 220 pag. della serie di fisica della « Collection Armand Colin » presenta l'acustica sotto questo profilo moderno, pur mantenendosi ad un livello di semplice introduzione e trattati più rigorosi e completi.

La suddivisione della materia trattata nei nove capitoli di cui l'opera si compone, appare del tutto logica. Dopo una prima parte (Cap. I) dedicata alla necessaria definizione delle grandezze acustiche, alla equazione della propagazioni e ad elementari nozioni di acustica fisiologica, si considerano in primo luogo, i sistemi a costanti concentrate (Cap. II) poi quelli a costanti distribuite (Cap. III) ponendo le basi dell'analogia con i circuiti elettrici corrispondenti.

Da tale analogia seguono in modo chiaro i fondamentali concetti di impedenza meccanica ed acustica.

Le applicazioni della nozione di impedenza alla riflessione e rifrazione delle onde sonore e allo studio delle onde stazionarie vengono trattate nel Cap. IV mentre i capitoli successivi sono dedicati alla propagazione nelle trombe esponenziali e coniche (Cap. V), ai vari tipi di microfono (Cap. VI-VII) all'irraggiamento del campo sonoro (Capitolo VIII) e ai più comuni tipi di altoparlanti elettrodinamici (Cap. IX). Tutte le volte che si tratta di studiare un elemento meccanico o elettroacustico, questo vien fatto mediante il suo circuito equivalente.

Pregio del volumetto di Rouard è

la chiarezza di esposizione ed il modo semplice con il quale i problemi applicativi vengono presentati.

A. Barone

W. H. Louisell – *Coupled Mode and Parametric Electronics.* John Wiley and Sons Inc. Publishers, New York, N. Y.

Questo libro ha lo scopo, raggiunto con molto successo, di presentare la teoria di funzionamento di molti sistemi fisici, in particolare dispositivi elettronici, in forma unificata come teoria di « modi accoppiati ».

L'Autore introduce la teoria dei modi accoppiati servendosi dei più semplici modelli meccanici: gli oscillatori armonici.

A partire dalle due equazioni Hamiltoniane di un semplice oscillatore armonico, l'Autore introduce le equazioni normali (tra le variabili normali) equivalenti alle equazioni Hamiltoniane di partenza ma caratterizzate dall'essere disaccoppiate: la soluzione di ognuna di queste due equazioni. che possono risolversi separatamente definisce un modo normale dell'oscillatore. Per estensione vengono quindi definiti i quattro modi normali di due oscillatori armonici accoppiati e vengono messi in luce gli scambi energetici tra i due oscillatori e la loro dipendenza dai coefficienti di accoppiamento. In appendice viene mostrato che la ricerca dei moduli normali si riconduce alla diagonalizzazione di una matrice che caratterizza il sistema lineare di equazioni differenziali del primo ordine considerato. Alcuni paragrafi che riconducono all'algoritmo studiato i problemi di propagazione di onde elettromagnetiche lungo una linea, concludono i primi due capitoli che hanno carattere introduttivo. Il resto del libro è dedicato all'applicazione dei metodi introdotti: 1) ai tubi per microonde ad onda progressiva (accoppiamento di modi di fascetti elettronici con modi elettromagnetici di linee di ritardo); 2) a circuiti risonanti accoppiati da elementi conservati variabili nel tempo (accoppiamenti paralleli); 3) ad accoppiamenti parametrici di onde veloci di acrica spaziale in fascetti elettronici; 4) ad amplificatori parametrici basati sul modo « ciclotrone » di fascetti elettronici in campo magnetico assiale (tubi amplificatori di Adler) ed infine 5) agli amplificatori parametrici a ferrite basati sulle proprietà giroscopiche del vettore magnetizzazione $\boldsymbol{M}$. Ciascuno dei citati argomenti, richiederebbe, per una trattazione esauriente, un libro a sè, ma il libro si giustifica per il metodo unitario di trattazione che determina efficaci sintesi mentali in chi già conosce i singoli argomenti. Il libro però può anche servire per la sua esemplare chiarezza e per i continui ragionamenti fisici che alleggeriscono il peso degli sviluppi analitici, quale prima lettura per chi intraprende lo studio di questi dispositivi ed una molto esauriente bibliografia aiuta in questo caso lo studioso ad approfondire i singoli argomenti. Concludendo si tratta di un'utile pubblicazione di cui raccomandiamo la lettura.

A. Gatti

P. T. Landsberg – *Thermodynamics, with Quantum Statistical Illustrations.* Vol. II della collezione « Monographs in Statistical Physics and Thermodynamics »; Interscience Publishers, New York and London; pagine 499, $ 14.50.

La termodinamica moderna, come tutti sanno, ebbe origine nel 1909 con una giustamente celebrata memoria di Costantino Carathéodory. Sebbene siano trascorsi da allora ben cinquantadue anni, sono state stranamente poche le

trattazioni della Termodinamica che si siano ispirate alle idee profonde e rinnovatrici dello scienziato greco. E ciò malgrado le esortazioni di vari fisici autorevoli, come ad es. MAX BORN.

Il presente libro di LANDSBERG è una di queste poche eccezioni e merita pertanto un caldo benvenuto. Esso costituisce un logico sviluppo e coronamento del punto di vista di Carathéodory, alla concezione del quale rimane assai più vicino che non l'articolo di FALK e JUNG del nuovo « *Handbuch der Physik* », il quale si propone una assiomatizzazione più radicale.

Nel primo centinaio di pagine del suo libro il LANDSBERG discute i tre principi basilari della Termodinamica. Seguono poi due lunghi capitoli, rispettivamente intitolati: Applicazioni fondamentali ed estensioni; Combinazioni di Termodinamica e Meccanica statistica. In essi vengono in particolare illustrati i seguenti argomenti: gas ideale classico e quantistico, radiazione nera e non nera, metodo delle probabilità di transizione, la Termodinamica come antesignana della Meccanica quantistica, ecc.

Seguono poi cinque Appendici dedicate all'esposizione di questioni concettuali (per es. l'App. C contiene una sommaria discussione della questione ergodica e della « master equation ») e di sviluppi matematici ausiliari (l'App. B per es. tratta il problema della riduzione dei pfaffiani a forma canonica), che per motivi vari l'Autore non ha ritenuto opportuno inserire direttamente nel testo. Numerosi problemi corredano i vari argomenti.

Un appunto che, a mio avviso, potrebbe essere mosso al presente libro è la scarsa compattezza, la quale nuoce alla limpidità dell'esposizione e scema il vigore delle argomentazioni.

Questa critica non infirma tuttavia, ovviamente, il valore sostanziale del libro. Convenientemente utilizzato, esso potrebbe fra l'altro servire ad ammodernare la nostra didattica universitaria. In particolare la parte generale potrebbe venire esposta, con gli adattamenti del caso, nel primo biennio delle nostre Facoltà di Scienze, abbandonando — o riducendo a brevissimi cenni storico-introduttivi — le ormai superate impostazioni tradizionali.

Per finire, dirò di un lieve senso di sorpresa che ho provato all'apertura di questo libro: nella bella « galleria di ritratti » dei grandi Maestri della Termodinamica non figurano LUDWING BOLTZMANN, MAX PLANCK e ALBERT EINSTEIN.

A. LOINGER